PH.D. THESIS

ON

GENERALIZED INVERSES OF MATRICES

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Soibam Nabachandra Singh

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CERTIFICATE

It is a pleasure to certify that **Sri Soibam Nabachandra Singh** *has worked under my supervision for the last two years (July 4, 1984- July 4, 1986) on the Topic "* **Generalized Inverses Of Matrices"** *in the Department of Mathematics, Manipur University, Imphal, Manipur (India) for the award of the degree of Doctor of Philosophy of this University. The thesis being submitted by him satisfies all the requirements of the regulations relating to the degree of this University.*

Dated 10th *July, 1986.*

A.K.Chatterjee

Associate Professor
Department of Mathematics
Manipur University
Canchipur

PREFACE

This Thesis is an attempt to extend and give some new results connected with "GENERALIZED INVERSES OF MATRICES". At the end of the thesis, we have made an attempt to extend the definition of semi-group which feels slightly less than the definition of group. We have named it as 'Metroid'. This opens a new direction of group theory where lot of works can be done.

The thesis consists of six chapters. Chapter 1 comprises a short historical account of generalized inverses of matrices, which is a compilation of the previous writings on the subject.

In chapter 2, we study the properties of transvection matrix covering upto reducibility and irreducibility. The decomposability and indecomposability of transvection matrix are also discussed. Using transvection matrix we obtain the generalized inverses (or g-inverses) of matrices over a division ring. We conclude with the reflexive g- inverse given in the form of theorem.

In chapter 3, we are concerned with the g-inverses which satisfy subsets of the six equations (1), (2), (3), (4), (1^k) and (5) extending upto the Drazin-pseudoinverse and the Group inverse of a square matrix. And we discuss these g-inverses for matrices of index zero or one, over a division ring by using transvection and orthogonal matrices.

In chapter 4, we study various types of g-inverses of matrices over the finite division ring. We obtain the general form of matrices which

have g-inverses. And we have also attempted to define a new dimension to the division ring and positive definite matrices over it and some results are obtained.

In chapter 5, we define a homomorphism on matrices over a finite division ring and some results in the form of theorems are obtained such as, for a homomorphism f and some matrices A and B over a finite division ring, $f(A) = f(B)$ if and only if $A = B$ and X is a g-inverse of A if and only if $f(X)$ is a g-inverse of $f(A)$ etc. Some of these results are extended over an arbitrary division ring. Further, the idea of anti-isomorphism of Jacobson, N. has been used to the semigroup of g-inverses of matrices over any arbitrary division ring.

In chapter 6, we study mainly that similar matrices (particularly singular) of the same order over a division ring form a semigroup with identity. And the g-inverses of singular matrices also form a semigroup. Using the technique of Greville, T.N.E. , we attempted that rectangular or singular matrices form a new dimension to group called 'Matroid' with idempotent matrix as identity and a few properties are also discussed thereon.

We have given our references in the literatures either as [2] or [4, p68] or4.2.3. The digit in a square bracket will stand for the serial number of our reference list given at the end of the text. The reference [4, p68] means the 68th page of the book, journal or paper of the 4th reference number. For the digits separated by the points, the first digit will denote the chapter number, the second digit will denote the section number of the chapter and the third digit will denote the theorem

number or the lemma number of that section.

All the references used by me are given at the end of the thesis in the alphabatical order e.g.

1. Drazin, M.P. (1958) : Pseudoinverses in associative rings and semigroups, Amer. Math. Monthly. 65, (506-514).

Surname coming first followed by initials, then the year of publication, name of the topic (or article) or book, name of the journal or book, then volume number and then the page number of the journal or book.

The whole research has been done by me in the department of Mathematics, Manipur University, Canchipur, Imphal during 1984-1986.

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In writing this thesis, I have consulted a number of books or papers on the subject, to all the authors of those books or papers, I express my grateful thanks.

Lastly, I express my gratitute to my family and friends who encourage me in doing research work.

S. Nabachandra Singh

Dated 7th July,'86. Soibam Nabachandra Singh

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CHAPTER 1

HISTORICAL INTRODUCTION

It is commonly known that if A is a square non singular matrix, then there exists a matrix X, such that $AX = XA = I$, then X is called the inverse of A and denoted by A^{-1}. If A is a singular or a rectangular matrix, no such matrix X exists.

Moore [42 and 43] extended the idea of inverse to a singular matrix. He defined an inverse of a singular matrix A as the matrix X satisfying the following conditions

$$AX = P_A \qquad (1)$$

$$XA = P_X \qquad (2)$$

where P_Y stands for the projection operator onto $M(Y)$, the space generated by the columns of Y.

Tseng [83, 84 and 85] floated the idea of inverses of singular operators (bounded and closed), which are more general than matrices.

Penrose [56] forwarded his own inverse, which can be stated as follows:

$$AXA = A \qquad (3)$$

$$XAX = X \qquad (4)$$

$$(AX)^* = AX \qquad (5)$$

$$(XA)^* = XA \qquad (6)$$

these equations have a unique solution X for every matrix A. This X is

called the generalized inverse of A.

Moore named it as "general reciprocal" whereas Draztn [19] called it as "pseudoinverse".

It was shown by Rado [62] that the two definitions of Moore (1) and (2) and Penrose (3), (4), (5) and (6) are equivalent. The Moore-Penrose generalized inverse is denoted as $X = A^{+}$.

Any matrix X which satisfies

$$AXA = A$$

and $$XAX = X$$

of Penrose's definition (3) and (4) is called a reflexive generalized inverse of A and is denoted as $X = A_r^{-}$.

Rao and Mitra [67,p20] stated that

$$X = A^{-}$$

is a generalized inverse of A such that $x = Xy$ is a solution of $Ax = y$ for any y which makes the equation consistent.

For example, if

$$A = \begin{bmatrix} 4 & -1 & -3 & 2 \\ -2 & 5 & -1 & -3 \\ 2 & 13 & -9 & -5 \end{bmatrix}$$

be a 3x4 matrix,

then $$A^{+} = \frac{1}{6398}\begin{bmatrix} 608 & -362 & 130 \\ -197 & 212 & 242 \\ -431 & 204 & -250 \\ 324 & -235 & -57 \end{bmatrix}.$$

The above frame work will lead us to find the generalised inverses of different matrices over a division ring and to discuss their properties. The properties are used in group theory.

2. Several generalisation of these inverses of singular matrices come forward from time to time from different Mathematicians. We state here a few of them which have been studied by us in the coming chapters.

3. Rao [68] and then Mitra [49 and 50] studied generalized inverse quite in detail. Rao's pseudoinverse satisfies only equation (3) of Penrose. Later, Rao called X as generalized inverse (g-inverse). A g-inverse so defined is not unique, and so become interesting for further studies.

Rao [67,p209] took a matrix A of order $m \times n$ of rank r and then if $A = BC^{*}$, where B is $m \times r$ and C is $r \times n$ and each of rank r, then

$$B^{+}_{MI} = (B^{*}MB)^{-1}B^{*}M$$

$$(C^{*})^{+}_{IN} = N^{-1}C(C^{*}N^{-1}C)^{-1}$$

$$A^{+}_{MN} = (C^{*})^{+}_{IN}B^{+}_{MI}. \qquad (7)$$

In particular,

$$B^{+} = (B^{*}B)^{-1}B^{*}$$

$$(C^{*})^{+} = C(C^{*}C)^{-1}$$

$$A^{+} = C(C^{*}C)^{-1}(B^{*}B)^{-1}B^{*}$$

$$= (C^{*})^{+}B^{+} \qquad (8)$$

where B^{+}_{MI} stands for minimum I-norm M-least squares g-inverse of

B and A^{+}_{MN} stands for minimum N-norm M-least squares g-inverse of A.

At the same time one can find non-singular matrices A and B such that $BAC = \Delta$, where Δ is a diagonal matrix, not necessarily square, with non-zero elements in the main diagonal. Then

$$A^{-} = C \quad \Delta^{-} \quad B \qquad (9)$$

is a reflexive g-inverse of A. Thus we can reduce the matrix in a diagonal form.

In Rao[67,p209], we find that when a square matrix A is considered, then a non-singular matrix C of the same order exists such that CA is in Hermite Canonical form and is idempotent. Then C is A^{-}.

If we consider a matrix A of order $m \times n$, it can be written in the form

$$A = \lambda_1 V_1 U_1^{*} + \ldots + \lambda_r V_r U_r^{*} \qquad (10)$$

where $\lambda_1^{\gamma}, \ldots, \lambda_r^{\gamma}$ are the non-zero eigenvalues of AA^{*} or $A^{*}A$ and the Vectors $V_1, \ldots, V_r$ are orthonormal eigenvectors of AA^{*} and $U_1, \ldots, U_r$ are orthonormal eigenvectors of $A^{*}A$ corresponding to the eigenvalues $\lambda_1^2, \ldots, \lambda_r^2$. Then it is verified that

$$A^{+} = \lambda_1^{-1} U_1 V_1^{*} + \ldots + \lambda_r^{-1} U_r V_r^{*} \qquad (11)$$

More generally, if M and N are given positive definite matrices, A can be expressed in the form

$$A = N^{-1}[\kappa_1 \S_1 j_1^{*} + \ldots + \kappa_r \S_r j_r^{*}]M \qquad (12)$$

where $\kappa_1^{\gamma}, \ldots, \kappa_r^{\gamma}$ are the non-zero eigenvalues of $AN^{-1}A^{*}$ with respect to

M^{-1} or of $A^{\#}MA$ with respect to N. The vectors ξ_i are eigenvectors of $AN^{-1}A^{*}$ with respect to M^{-1} and η_i are eigenvectors of $A^{*}MA$ with respect to N. Then

$$A^{+}_{MN} = \kappa_1^{-1}\eta_1\xi_1^{*} + \cdots + \kappa_r^{-1}\eta_r\xi_r^{*}. \qquad (13)$$

While Greville [22] states that an mxn matrix A of rank $r > 0$ can be expressed as a product

$$A = BC \qquad (14)$$

where B is mxr and C is rxn, and both are of rank r.

Then the g-inverse of A is given by

$$A^{+} = C^{t}(CC^{t})^{-1}(B^{t}B)^{-1}B^{t} \qquad (15)$$

where the superscript t denotes the transpose. For particular cases $m = r = n$, equation (14) reduces to $A = BI$ and $A = IC$ respectively, and equation (15) reduces to (14) and equation (2) i.e. $C^{+} = C^{t}(CC^{t})^{-1}$ of [23]. Therefore equation (15) can be written in the form

$$A^{+} = C^{+}B^{+}. \qquad (16)$$

Generalized inverses of matrices of maximal rank due to Greville [23] are given below:

Let B be an nxm matrix of rank m, where $m \leq n$. Since the columns of B are linearly independent, any vector $v = Bu$ vanishes only when u is a zero vector. Therefore,

$$u^{t}B^{t}Bu = v^{t}v > 0 \text{ whenever } u \neq 0.$$

Thus B^tB is positive definite and hence non-singular. Greville defines his g-inverse (or pseudoinverse) of B as

$$B^+ = (B^tB)^{-1}B^t. \qquad (17)$$

If $m = n$, then (17) reduces to ordinary inverse of a matrix. For $m < n$, the g-inverse is a left inverse of B. There are other left inverses, this is the only one having rows in the row-space of B^t.

In the same way Greville defined the g-inverse of an mxn matrix C of rank m as

$$C^+ = C^t(CC^t)^{-1}. \qquad (18)$$

This is the only right inverse of C having columns in the column-space of C^t.

As for a general matrix A of order mxn, ($m \leq n$) and of rank r, r may be less than its smaller dimension m. Let B denote a matrix of r columns whose columns form a basis for the column-space of A. In particular, B might be formed by selecting r linearly independent columns of A. In other cases, a basis might be known a priori from the manner in which A was derived. Similarly, let C denote an r-rowed matrix whose rows form a basis for the row-space of A. The g-inverses of B and C are then given by the formulas (17) and (18).

To find the g-inverses of A, Greville first noted that A has a unique left identity matrix with rows in the row-space of A^t. This was shown to be

$$I_L = BB^+. \qquad (19)$$

For, evidently $I_L B = B$ and

$$I_L A = A \qquad (20)$$

follow, since each column of A is a linear combination of columns of B. On the other hand, if I_L is of the form XB^t and satisfies equation (20), it preserves every vector in the column-space of A, and $XB^tB = B$.

Thus $X = B(B^tB)^{-1}$ and $I_L = BB^+$.

Similarly, $I_R = C^+C \qquad (21)$

is the only matrix with columns in the column-space of A^t which satisfies the relation

$$AI_R = A. \qquad (22)$$

Then, for any matrix A, there is a unique matrix A^+, which has its rows and columns in the row-space and column-space of A^+ and also satisfies the equations

$$AA^+ = I_L$$
$$A^+A = I_R. \qquad (23)$$

This matrix A^+ is called the g-inverse (or pseudoinverse) of A. Using these conditions, we have tried to develop 'Matroid' in chapter 6, something very similar to group.

Rao and Mitra [67,p62] have given Neumann type expansion for A^+. Let A be an $m \times n$ matrix and consider the representation $A = BDC^*$, where B and C are unitary matrices of order m and n, respectively

$$D = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$$

and $\Lambda = diag(\alpha_1 I_{r1}, \alpha_2 I_{r2}, \ldots, \alpha_u I_{ru})$, $\Sigma r_i = r$.

Then, $$A^+ = CD^+B^* \qquad (24)$$

Where, $$D^+ = \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and $\Lambda^{-1} = diag(\alpha_1^{-1} I_{r1}, \alpha_2^{-1} I_{r2}, \ldots, \alpha_u^{-1} I_{ru})$.

Rao and Mitra proved the following theorems:

(a) Let $0 < \alpha < 2/C$, *where* $C = max(\alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2)$.

Then $\alpha \sum_{k=0}^{\infty} (I - \alpha A^* A)^k A^*$ *converges and is equal to* A^+.

(b) With α *as in theorem a, let*

$$A_p^+ = \alpha [\{I + (I - \alpha A^* A)\} \prod_{k=1}^{p-1} \{I + (I - \alpha A^* A)^{2^k}\}] A^*$$

then

$$\| A^+ - A_p^+ \| \leq \max_i \frac{(1 - \alpha \alpha_i^2)^{2^p}}{\alpha_i} \qquad (25)$$

where for a matrix B, $\| B \|$ *represents the matrix norm defined by*

$$\| B \| = max \{ \sqrt{\lambda_i} : \lambda_i \in \sigma(B^* B), \text{ the spectrum of } B^* B \} .$$

(c) For a complex matrix A

(i) $$A^+ = \sum_{k=1}^{\infty} A^* (I + AA^*)^{-k}$$

$$= \sum_{k=1}^{\infty} (I + A^* A)^{-k} A^* \qquad (26)$$

(ii) $$A^+ = \lim_{\lambda \to 0+} A^* (\lambda I + AA^*)^{-1}$$

$$= \lim_{\lambda \to 0+} (\lambda I + A^* A)^{-1} A^* . \qquad (27)$$

Generalized inverses of partitioned matrices have been discussed by various Mathematicians like Greville [22], Cline [15] etc.

Greville's form for the generalized inverse of a matrix

$$A_k = [\, A_{k-1} \,,\; a_k \,]$$

can be stated as follows:

Let $d_k = A_{k-1}^{+} a_k$ and

$$c_k = a_k - A_{k-1}\, d_k \,.$$

Then

$$A_k^{+} = \begin{bmatrix} A_{k-1}^{+} - d_k\, b_k \\ b_k \end{bmatrix} \qquad (28)$$

where

$$b_k = \begin{cases} c_k^{+}, & \text{if } c \neq \Theta \text{ where } \Theta \text{ denotes the } m \text{ by } n \text{ null matrix} \\ (1 + d_k^{*} d_k)^{-1} d_k^{*} A_{k-1}^{+}, & \text{if } c_k = \Theta \,. \end{cases} \qquad (29)$$

This form for A_k^{+} was extended by Cline [15] to obtain representations for the g-inverse of matrices $A = [\, U \,,\; V \,]$, by combining (28) and (29) into a single expression. The single expression for b_k for the cases $c_k \neq \Theta$ and $c_k = \Theta$ in (29) is

$$b_k = c_k^{+} + (1 - c_k^{+} c_k)(1 + d_k^{*} d_k)^{-1} d_k^{*} A_{k+1}^{*} \,. \qquad (30)$$

combining (28) and (29) then gives

$$A_k^+ = \begin{bmatrix} A_{k-1}^+ - A_{k-1}^+ a_k c_k^+ - A_{k-1}^+ a_k (1-c_k^+ c_k) K_1 a_k^* A_{k-1}^{+*} A_{k-1}^+ \\ \\ c_K^+ + (1-c_k^+ c_k) K_1 a_k^* A_{k-1}^{+*} A_{k-1}^+ \end{bmatrix} \tag{31}$$

where K_1 designates the quantity $(1+d_k^* d_k)^{-1}$ and $a_k^* A_{k-1}^{+*}$ is utilized in place of d_k^*. The expression in (31) exhibits the structure of the representations for the generalized inverse of matrices $A = [U, V]$.

Cline [15] stated the g-inverse of any matrix $A = [U , V]$ in the following equivalent forms:

$$\text{(a)} \quad A^+ = \begin{bmatrix} U^+ - U^+ V C^+ - U^+ V (I-C^+ C) K V^* U^{+*} U^+ (I-V C^+) \\ \\ C^+ + (I-C^+ C) K V^* U^{+*} U^+ (I-V C^+) \end{bmatrix} \tag{32}$$

$$\text{(b)} \quad A^+ = \begin{bmatrix} U^+ - U^+ V C^+ - U^+ V (I-C^+ C) K V^* U^{+*} U^+ (I-V C^+) \\ \\ V^+ - V^+ U \tilde{C}^+ - V^+ U (I - \tilde{C}^+ \tilde{C}) \tilde{K} U^* V^{+*} V^+ (I-U \tilde{C}^+) \end{bmatrix} \tag{33}$$

$$\text{(c)} \quad A^+ = \begin{bmatrix} \tilde{C}^+ + (I - \tilde{C}^+ \tilde{C}) \tilde{K} U^* V^{+*} V^+ (I-U \tilde{C}^+) \\ \\ C^+ + (I-C^+ C) K V^* U^{+*} U^+ (I-V C^+) \end{bmatrix} \tag{34}$$

where $C = (I-UU^+)V$, $\tilde{C} = (I-VV^+)U$,

$K = [I+(I-C^+C)V^+U^{+*}U^+V(I-C^+C)]^{-1}$,

$\tilde{K} = [I+(I-\tilde{C}^+\tilde{C})U^*V^{+*}V^+U(I-\tilde{C}^+\tilde{C})]^{-1}$.

Rohde [61] extended the inverse of a partitioned matrix in terms of matrices of lower order to generalized inverses of partitioned matrices.

Rohde termed $X^{(r)}$, a reflexive semi-inverse, which obeys the relations

$$X^{(r)}XX^{(r)} = X^{(r)}$$

and $X^{(a)}$ a semi-inverse satisfying

$$XX^{(a)}X = X.$$

$X^{(N)}$ denotes a normalized generalized inverse or a weak generalized inverse obeying the relations

$$X^{(N)}XX^{(N)} = X^{(N)} \text{ and}$$

$$XX^{(N)} = [XX^{(N)}]^H ,$$

where H denotes the conjugate transpose and X^+ the generalized inverse or pseudoinverse.

Rohde [61,p1035] stated the g-inverses of non-negative Hermitian matrix in a theorem.

If a non-negative Hermitian matrix M is partitioned in the form

$$M = \left[\begin{array}{c:c} A & C \\ \hdashline C^H & B \end{array}\right]$$

then

(a) a g-inverse of M is given by

$$M^{(g)} = \left[\begin{array}{c|c} A^{(g)} + A^{(g)}CQ^{(g)}C^{H}A^{(g)} & -A^{(g)}CQ^{(g)} \\ \hline -Q^{(g)}C^{H}A^{(g)} & Q^{(g)} \end{array}\right] \qquad (35)$$

where $Q = B - C^{H}A^{(g)}C$,

(b) a reflexive g-inverse of M is given by (35) with $A^{(g)}$ and $Q^{(g)}$ replaced by $A^{(r)}$ and $Q^{(r)}$,

(c) a normalized g-inverse of M is given by (35) with $A^{(g)}$ and $Q^{(g)}$ replaced by $A^{(N)}$ and $Q^{(N)}$,

(d) a pseudoinverse of M is given by (35) with $A^{(g)}$ and $Q^{(g)}$ replaced by A^{+} and Q^{+}.

Various Mathematicians have studied the generalized inverses of integral matrices.

Batigne [9] states that, if A be a square, integral, Hermitian matrix, then

$$A^{+} = f(A) \qquad (36)$$

for some polynomial

$$f(x) = \frac{x+(\alpha_1^{-1}\alpha_2 x-1)\,\alpha_1^{-1}\mu(x)}{x^2}$$

and $\mu(x) = x^{t} + \alpha_{t-1}x^{t-1} + \ldots + \alpha_1 x.$

If the coefficient α_1 of x in $\mu(x)$ is equal to ± 1, then $f(x)$ is an integral polynomial and A^{+} is an integral matrix.

A matrix A is pseudo-regular if there exist permutation matrices P and Q such that

$$A = P \begin{bmatrix} M_A & 0 \\ 0 & 0 \end{bmatrix} Q$$

where M_A is non singular,

then

$$A^+ = Q^* \begin{bmatrix} M_A^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^* \qquad (37)$$

and A^+ is integral if and only if A ($\neq(0)$) is pseudo-regular and $\det(M_A) = \pm 1$.

Hall and I.J. Katz[29] studied mainly the integral $\{1,3\}$ -and $\{1,4\}$ -inverses of integral matrices.

The following statements were proved :

(a) A has an integral $\{1\}$ - inverse if and only if, the greatest common divisor (g.c.d) of all rxr subdeterminants of A is 1.

(b) A has an integral $\{1,3\}$ -inverse if and only if, the g.c.d of all rxr subdeterminants of A is 1 and has exactly r nonzero rows.

(c) A has an integral $\{1,4\}$ -inverse, if and only if the g.c.d of all rxr subdeterminants of A is 1 and A has exactly r nonzero columns.

(d) A^+ is integral if and only if, there exists a unique nonzero rxr subdeterminant of A and this subdeterminant is ± 1.

Thus we see that A has an integral $\{1,3\}$ -inverse ($\{1,4\}$ -inverse) if and only if A has exactly r nonzero rows (columns) these rows (columns) form a matrix which has an integal $\{1\}$ -inverse.

Many mathematicians like Pye [58], Hall and I.J. Katz [29], Chien-Fu Wu [18] etc, discussed non negative generalized inverses.

Pye [58] studied a necessary and sufficient condition for a real matrix to have a non negative Drazin inverse.

For a real matrix A, $P(A)$ and $N(A)$ denote its range and null space. When A is a square matrix, index A denotes the smallest non negative integer k for which $N(A^{k+1}) = N(A^k)$. If $k \geq$ index A is equivalent to each of the following:

$R(A^{k+1}) = R(A^k)$ and $R^n = R(A^k) + N(A^k)$, where $A \in R^{nxn}$.

The Drazin inverse of a square matrix A denoted $A^{(d)}$, is a solution for the system of equations

$$XAX = X$$

$$AX = XA$$

$$\text{and } A^{k+1}X = A^k . \qquad (38).$$

Pye called the Drazin inverse as group inverse and denoted them as $A^{\#}$.

The Drazine inverse of an nxn matrix A is nonnegative if and only if for some integer $k \geq 0$

$$Ax \in R_+^n + N(A^k) \text{ and } x \in R(A^k) \Rightarrow x \geq 0. \qquad (39)$$

Further, for a non negative nxn matrix A with index $A = k$ and rank $A^k = s$, A has a non negative Drazin inverse if and only if there exists a nonsingular matrix P such that

$$A = P \begin{bmatrix} M & 0 \\ 0 & p \end{bmatrix} P^{-1}$$

$$= [\, P_1 \quad P_2 \,] \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

$$= P_1 M Q_1 + P_2 N Q_2 \qquad (40)$$

where M is monomial, N is nilpotent, P_1 is a nonnegative nxs matrix, and Q_1 is nonnegative sxn matrix.

Hall and Katz [29] studied the necessary and sufficient conditions for a nonnegative integral matrix to have nonnegative integral generalized inverses of various types and the possible ranks of these inverses were also determined in the form of theorems.

If A be an mxn non negative integral matrix of rank r, then

(a) A has a non negative integral $\{1\}$ -inverse if and only if there exist permutation matrices P,Q such that

$$PAQ = \begin{bmatrix} I_r & * \\ * & * \end{bmatrix}$$

(b) A has a non negative integral {1,3} - inverse if and only if there exist permutation matrices P,Q such that

$$PAQ = \begin{bmatrix} I_r & * \\ 0 & 0 \end{bmatrix}$$

(c) A has a non negative integral {1,4} - inverse if and only if there exist permutation matrices P,Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ * & 0 \end{bmatrix}$$

(d) A^+ is non negative and integral if and only if there exist permutation matrices P,Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} .$$

Chien-Fu Wu [18] also studied the extensions of the result $A \geq B$ iff $B^{-1} \geq A^{-1}$, (where A and B are positive definite matrices) to any real non negative definite matrices A and B.

Thus for any two nxn non negative definite (n.n.d) matrices A and B, $A \geq B$ implies $A^- \leq B^-$.

If A and B are of ranks r and s, $A \geq B$, and $\bar{r}$, $\bar{s}$ be any two positive integers satisfying $\bar{s} \geq \bar{r} \geq r \geq s$, and for any n.n.d. $\{1\}$ - inverse $A^{(1)}$ of rank $\bar{r}$, there exists an n.n.d. $\{1\}$ - inverse $B^{(1)}$ of rank $\bar{s}$ such that $B^{(1)} \geq A^{(1)}$.

Rong, Guang Hao [72] stated that for a square matrix A and a sufficiently large positive integer k there is a unique matrix A^D, called the Drazin inverse of A, which satisfies the three matrix equations

$$A^{k+1} A^D = A^k$$

$$A^D A A^D = A^D$$

and $$A A^D = A^D A.$$

Let $B = A+E$ denote a perturbation of A. Then under certain conditions A^D is continuous (in the sense that $\{A_t\} \to A$ implies $\{A_t^D\} \to A^D$), and the author presented an upper bound for the relative error $\| B^D - A^D \|_2 / \| A^D \|_2$ and a lower bound for $\| B^D \|_2$.

Aitchison, P.W.[3] described the definitions and properties of some of the common generalized inverse matrices, including methods for constructing them. A number of applications were discussed and were illustrated by simple examples.

Initially, applications were not common, but with the passage of time, applications of generalized inverses have increased four fold. At present, we find it is used in Economics, Statistics, Functional analysis, Biological Sciences, System analysis, Network theory etc.

Mathematicians like Bott, R. and R.J.Duffin [13], Seshu, S. and M.B. Reed [82], Ben-Isreal, A. and A. charnes [8], sharpe, G.E. and G.P.H. Styan [80] and [81], Manherz, R.K. [54] etc. studied the applications of g- inverses in Network theory.

Rosen, J.B.[70] and [71], Cline, R.E. [17], pyle, L.D. [59], Ben-Israel, A and A. charnes [7] etc. worked on mathematical programming.

Rao, C.R. [65] and [66], Anderson, C.L. [1], Rohde, C.A. [60], Graybel, F.A. [26] etc. studied special applications of the theory of generalized matrix inversion to statistics.

Applications of generalized matrix inversion to the solution of systems of equations and Functional analysis were discussed by Greville, T.N.E. [25] and [23], Vortuba, C.[86], Ben- Israel, A. [6], Khatri, C.G. and C.R Rao [40], Neudecker, H. [55], Mitra, S.K. [51] and [52], Bhimasankaran, P. ; Majumdar, Dibyen, [12] etc.

Generalized inverses of matrices were used to solve least square problems by mathematicians like Albert, A. and R.W. Sittler [2], Golub, G. [27] and [28], Barreto, G. and Jose, A. [14], etc.

Some authors such as Lewis, T.O. [41], Marquardt, D.W. [53], Hall, F.J.; Mayer, Carl D., Jr. [36] etc. examined the use of g- inverses of matrices in linear estimation problems.

And some other scholars like Searle, S.R. [75], Shanbag, D.N. [78] and [79] etc. used the theory of generalized matrix inversion to Biological Sciences.

CHAPTER 2

TRANSVECTION MATRICES EQUIVALENT TO IRREDUCIBLE MATRICES AND ITS APPLICATION TO GENERALIZED INVERSES OF MATRICES OVER DIVISION RING[*] .

Introduction. In this chapter, firstly we have to discuss the properties of a transvection matrix over a division ring. The discussion covers upto reducibility, irreducibility, decomposability and indecomposability including fully indecomposability and partly decomposability of a transvection matrix. Secondly, using transvection matrix we obtain the generalized inverses of matrices over a division ring. And we conclude with the reflexive generalized inverse given in the form of theorem 1.

Definition 1. Let λ be a non-zero element on a division ring $\mathbb{D}$ and $i \neq j$ integers between 1 and n. We recall [64,p157] that a transvection $T_{ij}(\lambda)$ is an nxn matrix differing from the identity matrix I in that it has λ as its ijth entry. If I_{ij} is the nxn matrix with 1 in the ijth place and 0 elsewhere, then

$$T_{ij}(\lambda) = I + \lambda I_{ij} \qquad (1)$$

Thus $T_{12}(\lambda) = \begin{bmatrix} 0 & \lambda \\ 0 & 1 \end{bmatrix}$, $T_{21}(-2\lambda) = \begin{bmatrix} 1 & 0 \\ -2\lambda & 1 \end{bmatrix}$

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[*] Appeared in "The Mathematics Education" (Bihar, India) Vol.XVII, NO.2, June 1983, p(77-82).

$$T_{23}(\mu) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{bmatrix}, \quad T_{31}(-2\mu) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2\mu & 0 & 1 \end{bmatrix}.$$

where $\mu \quad (\neq 0) \in \mathbb{D}$, are few examples of transvection matrices of second and third order.

2. Some Properties of Transvection Matrix

(i) Since a transvection matrix $T_{ij}(\lambda)$ can be derived from a unit matrix I_n by a chain of elementary operations (or E-operations), $T_{ij}(\lambda)$ is equivalent to I_n. Therefore $T_{ij}(\lambda)$ can be expressed as a product of elementary matrices i.e. $T_{ij}(\lambda) = E_1 \cdot \; \cdot \; \cdot E_k \, E^{(1)} \cdot \; \cdot \; \cdot \; E^{(l)}$ where E_i and $E^{(j)}$ are suitable E-matrices.

(ii) The product of two transvection matrices is not a transvection matrix. For, if $T_{13}(\lambda)$ and $T_{23}(\mu)$ where non zeros $\lambda, \mu \in \mathbb{D}$ be two transvection matrices, then

$$T_{13}(\lambda) \cdot T_{23}(\mu) = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{bmatrix}$$

It is not a transvection matrix.

(iii) A transvection matrix can not represent a permutation matrix on n symbols.

In S_3, the symmetric group on three symbols 1, 2, 3 the six permutations are

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 33 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} .$$

And the corresponding permutation matrices are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

These are not transvection matrices .

(iv) For a transvection matrix $T_{ij}(\lambda)$, $T_{ij}(-\lambda)$ is the inverse of $T_{ij}(\lambda)$ such that $T_{ij}(\lambda).T_{ij}(-\lambda) = I = T_{ij}(-\lambda).T_{ij}(\lambda)$.

Definition 2. An nxn matrix A of 0's and 1's is reducible if there exists a transvection matrix $T_{ij}(\lambda)$ where $i \neq j$ are intejers between 1 and n, and $\lambda (\neq 0) \in \mathbb{D}$ the division ring such that

$$T_{ij}(\lambda) . A . T_{ij}^{t}(\lambda) = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} . \qquad (2)$$

where t stands for transpose of a matrix, 0 is a px(n-p) zero matrix for some integer p with $1 \leq p \leq n-1$, A_1 is a pxp matrix, A_{21} is (n-p)xp and A_2 is (n-p)x(n-p). A square matrix is irreducible if it is not reducible. It follows that a 1x1 matrix is irreducible.

For $\alpha, \beta \subseteq \{1, \ldots, n\}$, *let* $T[\alpha, \beta]$ *denote the submatrix of* T *determined by rows* i *with* $i \in \alpha$ *and columns* j *with* $j \in \beta$. *Then* T *is irreducible if and only if there does not exist a partition* α, β *of* $\{1, \ldots, n\}$ *into non empty sets such that* $T[\alpha, \beta]$ *is a zero matrix.*

For our purpose the following lemmas are useful.

Lemma 1. *An nxn transvection matrix is irreducible if and only if the px(n-p) zero submatrix for some integer* p *with* $1 \leq p \leq n-1$, *contains an entry from its main diagonal and it is reducible if and only if the px(n-p) zero submatrix does not contain an entry from the main diagonal.*

Proof. *First suppose that* T *is reducible. Then there exists a partition* α, β *of* $\{1, \ldots, n\}$ *into non-empty sets such that* $T[\alpha, \beta] = 0$. *Since* α, β *is a partition,* $T[\alpha, \beta]$ *contains no entry from a position of the main diagonal of* T *and* $|\beta| = n - |\alpha|$. *Similarly the converse is proved.*

Definition 3. *An nxn matrix* A *of 0's and 1's is partly decomposable if either* $n = 1$ *and* $A = [0]$, *or* $n > 1$ *and there exist transvection matrices* $T_{ij}(\lambda)$ *and* $T_{ij}(\mu)$ *where* $i \neq j$ *are integers between 1 and n and* λ $(\neq 0)$, μ $(\neq 0) \in \mathbb{D}$ *a division ring such that*

$$T_{ij}(\lambda) \cdot A \cdot T_{ij}(\mu) = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}, \qquad (3)$$

where 0 is a px(n-p) zero submatrix of A for some $p = 1, \ldots, n-1$.

And A is fully indecomposable if it is not partly decomposable.

Then transvection matrix $T_{ij}(1)$ *is both partly decomposable and fully indecomposable.*

Example : $T_{23}(\lambda) \cdot T_{21}(1) \cdot T_{31}(\mu)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mu & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1+\lambda\mu & 1 & 0 & 0 \\ \mu & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}.$$

Thus $T_{21}(1)$ *is partly decomposable.*

Also, $T_{23}(\lambda) \cdot T_{21}(1) \cdot T_{31}(\mu)$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1+\lambda\mu & 1 & \lambda & 0 \\ \hline \mu & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \neq \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix},$$

hence $T_{21}(1)$ *is fully indecomposable .*

Lemma 2. *Let A be an nxn matrix of 0's and 1's and T be a transvection matrix of the same order. Then A is irreducible if and only if T is fully indecomposable.*

Proof : The transvection T has a main diagonal of elements 1 taken from A and the other non zero element of T is taken from a division ring .

Here, A is irreducible if and only if each $px(n-p)$ zero sub-matrix of A with $1 \leq p \leq n-1$ contains an entry from the main diagonal of A (by lemma 1). The latter holds if and only if T has no $px(n-p)$ zero submatrix with $1 \leq p \leq n-1$, that is, if and only if T is fully indecomposable.

Lemma 3. *Let A be an nxn matrix of 0's and 1's and T be a transvection matrix of the same order. Then TA is irreducible if and only if A has row (or column) all of whose entries are zero or not all zero.*

Proof,: Suppose the first row of A is zero, then TA is a matrix of the same order and its first row is also zero. Then by the definition 2 , it is found that TA is irreducible .

Example: Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

be a 4x4 matrix .

Then, $T_{24}(3) \cdot A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = B \text{ (say)}$$

and $T_{32}(2)\ B\ T^{t}_{32}(2)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 4 & 11 & 0 \\ 3 & 6 & 22 & 1 \\ 0 & 1 & 3 & 0 \end{bmatrix} \neq \begin{bmatrix} B_1 & 0 \\ B_{21} & B_2 \end{bmatrix}$$

hence $TA = B$, is irreducible .

In the same way, if the first row of A is not all zero, then TA is also a matrix of the same order and its first row is not all zero. By definition 2, TA is irreducible.

Example : Let $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

be a square matrix of order four .

Then, $T_{24}(2) \cdot A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = B \text{ (say)}$$

and $T_{23}(2).\ B.T_{23}^{t}(2) =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 3 & 7 & 2 & 2 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 3 & 1 & 0 \end{array}\right] = \begin{bmatrix} B_1 & 0 \\ B_{21} & B_2 \end{bmatrix} .$$

3. Generalized Inverses Of Matrices Over a Division Ring.

In this section we shall be considering generalized inverses (or g-inverse-s) of matrices over a division ring.

Let A be an $m \times n$ matrix. An $n \times m$ matrix X is called a g-inverse of A, if, whenever $Ax = y$ has a solution, Xy is a solution of $Ax = y$. Since $\mathbb{D}$ has 1 (more generally, $\mathbb{D}$ has an element a such that $x.a = 0$ only when $x = 0$) it is easily seen that X is a g-inverse of A if and only if

$$A X A = A. \qquad (4)$$

We, further consider the following equations

$$X A X = X \qquad (5)$$

$$(AX)^* = AX \qquad (6)$$

$$(XA)^* = XA \qquad (7)$$

where B^* stands for the conjugate transpose of B. If X satisfies (4) and (5), it is called a reflexive g-inverse of A and is denoted by A_r^-, if it satisfies all the four equations it is called a Moore-penrose inverse of A. The Moore-penrose inverse when it exists is unique and is denoted by A^+.

Before studying the main theorem we establish some lemmas.

Smith normal form 4.

Due to [45,p26] every $m \times n$ matrix A of rank r can be written as

$$U \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} V$$

where U and V are units, $S = \text{diag}(s_1, s_2, \ldots, s_r)$, s_i's are non zero elements of $\mathbb{D}$ and $s_i \mid s_{i+1}$ for $1 \leq i \leq r-1$.

Lemma 4. A matrix $\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$ where $S = \text{diag}(s_1, s_2, \ldots, s_r)$ and $s_i \neq 0$ has a g-inverse if and only if s_i's are all non zero elements of $\mathbb{D}$. Further X is a g-inverse of $\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$ if and only if $X = \begin{bmatrix} S^{-1} & B \\ C & D \end{bmatrix}$ where $S = \text{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_r^{-1})$ and B, C and D are some matrices.

Proof : Let $\begin{bmatrix} E & B \\ C & D \end{bmatrix}$ be a g-inverse of $\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$, using equation (4) we get that $SES = S$. Therefore $|S|^2 |E| = |S|$, Since $|S| \neq 0$, $|S||E| = 1$ i.e. the product $s_1 s_2 \ldots s_r$ is a unit of $\mathbb{D}$. This implies that the s_i's are units of $\mathbb{D}$. In such case S^{-1} exists and $E = S^{-1}$.

Lemma 5. An $m \times n$ matrix A of rank r has a g-inverse if and only if $A = T_{ij}(\lambda) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(\mu)$ where $T_{ij}(\lambda)$ and $T_{ij}(\mu)$ are transvection matrices and I is the $r \times r$ identity matrix. Further an $n \times n$ matrix X is a g-inverse of A if and only if $X = T_{ij}(-\mu) \begin{bmatrix} I & B \\ C & D \end{bmatrix} T_{ij}(-\lambda)$ for some matrices B, C and D.

Proof : Let $A = T_{ij}(\lambda) \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(\mu)$ be a representation given by

Smith normal form. Then clearly, X is a g-inverse of A if and only if $T_{ij}(\mu)\,XT_{ij}(\lambda)$ *is a g-inverse of* $\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$ *and Y is a g-inverse of* $\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$ *if and only if* $T_{ij}(-\mu)\,Y\,T_{ij}(-\lambda)$ *is a g-inverse of A. Combining these two statements with the lemma 4 gives our result.*

Lemma 6. *Let* $A = T_{ij}(\lambda)\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}T_{ij}(\mu)$ *where* $T_{ij}(\lambda)$ *and* $T_{ij}(\mu)$ *are transvection matrices. Then X is a reflexive g-inverse of A if and only if it is of the form* $T_{ij}(-\mu)\begin{bmatrix} I & B \\ C & CB \end{bmatrix}T_{ij}(-\lambda)$ *for some matrices B and C.*

Proof : *By lemma 5 a g-inverse of A is of the form* $T_{ij}(-\mu)\begin{bmatrix} I & B \\ C & D \end{bmatrix}T_{ij}(-\lambda)$ *for some matrices B,C and D. If this is to be a reflexive g-inverse,* $\begin{bmatrix} I & B \\ C & D \end{bmatrix}$ *should be a reflexive g-inverse of* $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

This implies that D = CB.

Theorem 1. *Let the division ring* $\mathbb{D}$ *satisfy the condition* $d_1 = d_1^2 + d_2^2 + \ldots + d_n^2$ *implying that* $d_2 = d_3 = \ldots = d_n = 0$. *Then for an nxn matrix A of rank r, the reflexive g-inverse* A_r^- *of A exists if and only if there exist transvection matrices* $T_{ij}(\lambda)$ *and* $T_{ij}(\mu)$ *where* $\lambda\,(\neq 0)$, $\mu\,(\neq 0) \in \mathbb{D}$ *and a non-singular matrix N(A) of order r such that*

$$A = T_{ij}(\lambda) \begin{bmatrix} N(A) & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(\mu)$$

and in this case

$$A_P^- = T_{ij}(-\mu) \begin{bmatrix} N(A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(-\lambda) \ .$$

Proof. *From a condition on* $\mathfrak{D}$, *any idempotent matrix* M *is an* $n \times n$ *matrix which is expressed in the form* $\begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} I & 0 \\ -P & 0 \end{bmatrix}$. *For our convenience* $\begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix}$ *is taken as the form of* M *where* I *is the* $r \times r$ *identity matrix and* P *is a* $r \times (n-r)$ *matrix. Let there exist a transvection matrix* $T_{ij}(\lambda)$ *such that*

$$T_{ij}(-\lambda)\, M\, T_{ij}(\lambda) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

The existence of such $T_{ij}(\lambda)$ *is verified by the following example.*

Example 1. *Let* $T_{13}(\lambda) = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $M = \begin{bmatrix} 1 & 0 & -\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

where $P = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}$.

Then, $T_{13}(-\lambda)\, M\, T_{13}(\lambda)$

$$= \begin{bmatrix} 1 & 0 & -\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Let A be an nxn matrix for which A_r^- exists.

Let $A = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V$, *where U and V are non-singular matrices. Since* A_r^- *satisfies the equation* $XAX = X$, *and* A_r^- *is of the form* $V^{-1} \begin{bmatrix} I & B \\ C & CB \end{bmatrix} U^{-1}$ *by lemma 6. for some matrices B and C .*

Here $A_r^- = U \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} U$, *which is of the form* $\begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix}$

It can be shown by the following example.

Example 2. Let $U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$

then $U \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} U^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc} I & -P \\ 0 & 0 \end{array}\right] .$$

Then there exists a transvection matrix $T_{if}(\lambda)$ *such that*

$$T_{if}(-\lambda)\, U \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} U^{-1}\, T_{if}(\lambda) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad (a) .$$

Let $T_{if}(-\lambda)\, U = Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$ *where* Z *is unit,*

and $U^{-1}\, T_{if}(\lambda) = Z^{-1} = \begin{bmatrix} Z^{(1)} & Z^{(2)} \\ Z^{(3)} & Z^{(4)} \end{bmatrix}$.

Substituting in relation (a), we get

$$\begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} = Z^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Z = \begin{bmatrix} I & Z^{(1)}Z_2 \\ Z^{(3)}Z_1 & Z^{(3)}Z_2 \end{bmatrix} ,$$

It implies that $Z^{(3)} [\, Z_1 \quad Z_2] = 0$.

Since $[\, Z_1 \quad Z_2\,]$ *is non-singular, then* $Z^{(3)} = 0$.

So $Z_3 = 0$ *and hence* $T_{if}(-\lambda)\, U = \begin{bmatrix} Z_1 & Z_2 \\ 0 & Z_4 \end{bmatrix}$ *(b).*

It can also be verified by the following example .

Example 3 .

$$T_{13}(-\lambda)\, U = \begin{bmatrix} 1 & 0 & -\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 1-\lambda \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} ,$$

where $Z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Z_2 = \begin{bmatrix} 1-\lambda \\ 0 \end{bmatrix}$, $Z_3 = [\,0 \quad 0\,]$

and $Z_4 = [\,1\,]$.

Similarly, there exists another transvection matrix $T_{ij}(\mu)$ such that,

$$VT_{ij}(-\mu) = \begin{bmatrix} K_1 & 0 \\ K_3 & K_4 \end{bmatrix} , \qquad (c)$$

for some matrices K_1 , K_3 and K_4.

This part is also verified by the following example.

Example 4. $V\,T_{31}(-\mu) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\mu & 0 & 1 \end{bmatrix}$

$$= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1-\mu & 0 & 1 \end{array}\right] = \begin{bmatrix} K_1 & 0 \\ K_3 & K_4 \end{bmatrix}$$

we have $A = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V$

$$= T_{ij}(\lambda) \begin{bmatrix} Z_1 & Z_2 \\ 0 & Z_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 & 0 \\ K_3 & K_4 \end{bmatrix} T_{ij}(\mu)$$

by using (b) and (c)

$$= T_{tf}(\lambda) \begin{bmatrix} Z_1K_1 & 0 \\ 0 & 0 \end{bmatrix} T_{tf}(\mu) .$$

Since Z_1 and K_1 are non-singular, Z_1K_1 is also non-singular.

Thus we have $A = T_{tf}(\lambda) \begin{bmatrix} N(A) & 0 \\ 0 & 0 \end{bmatrix} T_{tf}(\mu)$,

where $N(A) = Z_1K_1$. when A is of this form, it can be verified that

$$A_r^- = T_{tf}(-\mu) \begin{bmatrix} N(A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} T_{tf}(-\lambda) ,$$

for, $A_r^- A A_r^- = A_r^-$ is satisfied .

Remark. In the above theorem 1 we find only the reflexive g-inverse of the square matrix A, but cannot find the Moore-Penrose inverse of A, because the equation $(AX)^t = AX$ is not satisfied by the expression

$$A = T_{tf}(\lambda) \begin{bmatrix} N(A) & 0 \\ 0 & 0 \end{bmatrix} T_{tf}(\mu)$$

of the theorem .

CHAPTER 3

GENERALIZED INVERSES OF MATRICES OF INDEX ZERO OR ONE OVER A DIVISION RING.*

Introduction And Notations :

We shall let C^{mxn} denote the mxn complex matrices over a division ring $\mathbb{D}$ and C_r^{mxn} the same with rank r.

For any $A \in C^{mxn}$, we define the index of A or the ind A, as the smallest non negative integer k such that rank A^k = rank A^{k+1}.

For an mxn Complex matrix, consider the matrix equations

$$A X A = A \qquad (1)$$

$$X A X = X \qquad (2)$$

$$(AX)^* = AX \qquad (3)$$

$$(XA)^* = XA \qquad (4)$$

and if $m = n$

$$A^k X A = A^k \qquad (1^k)$$

for some integer $k > 1$

$$AX = XA \qquad (5)$$

where X is an nxm complex matrix. The unique matrix X satisfying all the four equations (1), (2), (3) and (4) is the Moore-Penrose generalized inverse of A and is denoted by A^+.

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* Appeared in the "Mathematics Education" (Bihar, India) Vol. XVIII No. 4, Dec. 1984, p. 123-128.

We shall be concerned with generalized inverses (or g-inverses) which satisfy subsets of the six equations (1), (2), (3), (4), (1^k) and (5). Using Ben-Israel [5]'s notations, any $X \in C^{n\times m}$ which satisfies equations (i), (j), . . .,(l) from among the equations (1), (1^k), (2), (3), (4), and (5). is called an $\{i, j, \ldots, l\}$ -inverse of A. The set of $\{i, j, \ldots, l\}$ -inverses of A is denoted by $A^{(i,j,\ldots,l)}$.

For any $A \in C^{m\times n}$, $A^{(1,2,3,4)}$ is non empty and unique, $A^{(1,2,3,4)} = A^{+}$, the Moore-Penrose g-inverse of A .

For any $A \in C^{n\times n}$, the unique $A^{(1^k,2,5)}$ is non empty and if $k \geq \text{ind } A$, $A^{(1^k,2,5)} = A^{(d)}$, the Drazin pseudoinverse of A.

For any $A \in C^{n\times n}$, if ind A = 0 or 1, the unique $A^{(1,2,5)}$ is non empty and is known as the Group Inverse of A and is denoted by A.

Any $A \in C_r^{n\times n}$ is an Epr matrix if and only if $A^{(1,2,3,4,5)}$ is non empty, i.e., $A^{+} = A^{\#}$.

This chapter deals with the g-inverses of matrices $A \in C^{n\times n}$ with ind A=0 or 1 i.e. rank A = rank A^2 .

The generalized inverses of matrices $A \in C^{n\times m}$ with ind A = 0 or 1 are given in the following lemmas and theorems .

Ben-Israel [5,p119] has shown in a theorem that for a matrix $A \in C^{n\times n}$, rank A = rank A^2 if and only if $\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} A$ exists and

is equal to $AA^{\#}$.

Lemma 1. *Let A be an nxn real matrix of rank r, such that*

$$A = T_{ij}(\lambda) \begin{bmatrix} N(A) & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(-\lambda)$$

where N(A) is non-singular matrix of rank r and $\lambda \ (\neq 0) \in \mathbb{D}$ *then by theorem 1 of chapter 2, the reflexive g-inverse of A is given by*

$$A_r^{-} = T_{ij}(\lambda) \begin{bmatrix} N(A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(-\lambda) \ .$$

Definition 1. *According to Mirsky [44,p261 and p262] a Matrix Group is a set of matrices which form a group with respect to matrix multiplication.*

The matrix group must consist of square matrices of the same order and all these matrices are either non singular or else all singular.

The set of matrices $A(x) = \begin{bmatrix} x & \overline{x} \\ x & x \end{bmatrix}$ *where* $x \neq 0$, *is a matrix group, with A(1 /2) as its unit element and A(1/4x) as the inverse of A(x).*

A Group Matrix is a matrix which is an element of at least one matrix group [44,p273]. We use theorem (9.4.2) of [44,p273] which stated as follows:

The following four statements are equivalent

(i) A is a group matrix (singular),

(ii) rank A = rank A^2,

(iii) A is singular to a matrix of the form $\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$

where A_1 is non-singular square matrix,

(iv) rank $A = n - m_0(A)$, where $m_0(A)$ denote the number of zero characteristic roots of A.

This theorem states that the index of a group matrix A is either 0 or 1, and by (ii) we can write A as

$A = S^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} S$, where S is non-singular matrix.

Lemma 2. *If A be a group matrix, we can write*

$$A = T_{ij}(-\lambda) \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(\lambda),$$ where $T_{ij}(\lambda)$ is a transvection matrix.

Then by lemma 1, $$X = T_{ij}(-\lambda) \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(\lambda),$$

where X is the g-inverse of A.

Here, $$AX = T_{ij}(-\lambda) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(\lambda),$$

$$AXA = T_{ij}(-\lambda) \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} T_{ij}(\lambda) = A,$$

$$XA = T_{ij}(-\lambda)\begin{bmatrix} I & \overline{0} \\ \underline{0} & \underline{0} \end{bmatrix} T_{ij}(\lambda) = AX$$

and

$$XAX = T_{ij}(-\lambda)\begin{bmatrix} A_1^{-1} & \overline{0} \\ \underline{0} & \underline{0} \end{bmatrix} T_{ij}(\lambda) = X .$$

X satisfies (1),(2),(5), hence X is the group inverse of A and

$$A^{\#} = T_{ij}(-\lambda)\begin{bmatrix} A_1^{-1} & \overline{0} \\ \underline{0} & \underline{0} \end{bmatrix} T_{ij}(\lambda).$$

Moreover, when k = 1, X satisfies (1^k)*,(2),(5), hence* $X = A^{(1^k,2,5)} = A^{(d)}$ *the Drazin pseudoinverse of A, and it is also the group inverse of A, hence when k = 1,* $A^{(d)} = A^{\#}$.

Further if Q be an orthogonal matrix over $\mathbb{D}$ *, we can write*

$$A = Q^{-1}\begin{bmatrix} A_1 & \overline{0} \\ \underline{0} & \underline{0} \end{bmatrix} Q \quad (\text{Orthogonally similar})$$

(*If Q is orthogonal* $Q^tQ = I = QQ^t$ *and* $Q^t = Q^{-1}$)

$$= P\begin{bmatrix} A_1 & \overline{0} \\ \underline{0} & \underline{0} \end{bmatrix} Q, \text{ where } PQ = I.$$

Then we establish the following theorem.

Theorem 1. *Let A be an nxn matrix of rank r over a division ring* $\mathbb{D}$ *. Then* A^+ *exists if and only if there exist orthogonal matrices P and Q and a non singular matrix* A_1 *of order r such that*

$$A = P\begin{bmatrix} A_1 & \overline{0} \\ \underline{0} & \underline{0} \end{bmatrix} Q.$$

Then, $A^+ = Q^t \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^t$, where t stands for transpose of a matrix.

Proof. Let A be a group matrix over $\mathbb{D}$ for which A^+ exists.

Let $A = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V$, where U and V are units. Since A^+ satisfies equation (2) i.e. $XAX = X$, it is of the form $V^{-1} \begin{bmatrix} I & B \\ C & CB \end{bmatrix} U^{-1}$, by lemma 6 of chapter 2 for some matrices B and C.

Here, $AA^+ = U \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} U^{-1}$, which is an idempotent matrix.

Then there exists an orthogonal matrix P such that

$$P^{-1}AA^+P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$ (Theorem 1, relation (a) of chapter 2)

or $$P^{-1}U \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} U^{-1}P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \qquad (a)$$

Let $P^{-1}U = E = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}$, where E_1 is unit,

and $$U^{-1}P = E^{-1} = \begin{bmatrix} E^{(1)} & E^{(2)} \\ E^{(3)} & E^{(4)} \end{bmatrix}.$$

Substituting in (a), we have

$$\begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} = E^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} E = \begin{bmatrix} I & E^{(1)}E_2 \\ E^{(3)}E_1 & E^{(3)}E_2 \end{bmatrix}.$$

We get $E^{(3)} [\, E_1 \quad E_2 \,] = 0$.

Since $[E_1 \quad E_2]$ is non singular $E^{(3)} = 0$, then $E_3 = 0$ and hence

$$P^{-1} U = \begin{bmatrix} E_1 & E_2 \\ 0 & E_4 \end{bmatrix}.$$

Similarly there exists another orthogonal matrix Q, where $PQ = I$ such that

$$VQ^{-1} = \begin{bmatrix} F_1 & 0 \\ F_3 & F_4 \end{bmatrix} \text{ for some matrices } F_1, F_3 \text{ and } F_4.$$

Here, $A = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V$

$$= P \begin{bmatrix} E_1 & E_2 \\ 0 & E_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 & 0 \\ F_3 & F_4 \end{bmatrix} Q$$

$$= P \begin{bmatrix} E_1F_1 & 0 \\ 0 & 0 \end{bmatrix} Q.$$

Since E_1 and F_1 are units, E_1F_1 is also unit.

Thus $A = P \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} Q$, where $E_1F_1 = A_1$,

then it can be verified that

$$A^{+} = Q^{t} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{t}, \text{ for } (AA^{+})^{t} = AA^{+} \text{ is satisfied.}$$

Result 1. *For an nxn group matrix A of rank r ≤ 1, A^{+} satisfies equations (1),(2),(3),(4) and (5). Hence $A^{+} = A^{\#}$ and therefore A is an Epr matrix.*

Result 2. *The non singular matrix A_1 is unique.*

If possible let $A = P \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} Q = R \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix} S,$

where P,Q,R and S are orthogonal matrices and A_1 and A_2 are two non-singular matrices. Then we can show that $A_2 = TA_1U$ for T and U orthogonal.

Result 3. *If $A_1 = I_r$, such that* $A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q$, *then we write,*

(i) $A = P \begin{bmatrix} K \\ 0 \end{bmatrix}$, *where K is rxn matrix of rank r and 0 is (n-r)xn,*

(ii) $A = [H \quad 0]\, Q$, *where H is an nxr matrix of rank r and 0 is nx(n-r).*

Example. If A is a 3x3 matrix of rank 2, and

$$Q = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}, \text{ a 3x3 orthogonal matrix,}$$

$$\text{then } A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}$$

$$= P \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \end{bmatrix} = P \begin{bmatrix} K \\ \cdots \\ 0 \end{bmatrix}, \text{ where}$$

$$K = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$$

is a 2x3 matrix and 0 = [0 0 0] is 1x3.

Definition 2. *It is well known that if A and B are two matrices such that AB = I the identity matrix, then A is called the left inverse of B and B is called the right inverse of A.*

Corollary 1. *If A be an mxn matrix of rank m, then,*

(i) A has a right inverse,

(ii) A has a g-inverse and

(iii) there is a matrix B such that $\begin{bmatrix} A \\ B \end{bmatrix}$ *is unit.*

Proof. *Let AXA = A, since A is of full row rank there is a matrix B over* $\mathbb{D}$ *such that AB = I. Hence we have AXAB = AB or AX = I. Therefore A has a right inverse. And for a matrix A of full row rank a matrix X is a right inverse if and only if it is a g-inverse of A.*

For (iii), let A has a g-inverse. Then let A = U[I 0]V where U and V are units.

The matrices $\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}$, $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ and V are units and

$$\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} V = \begin{bmatrix} A \\ [\,0 \quad I\,]\,V \end{bmatrix},$$ which is unit.

If $[R_1 \quad R_2]$ is an inverse of $\begin{bmatrix} A \\ \cdots \\ B \end{bmatrix}$ then R_1 is a right inverse of A.

We define that for a square matrix A, a matrix which satisfies equations (1) and (3) is called a $\{1,3\}$ g-inverse and is denoted by A^-_{13}. Similarly A^-_{14} is also defined.

Theorem 2 : Let the division ring $\mathbb{D}$ satisfy the condition $d_1 = d_1^2 + d_2^2 + \ldots + d_n^2$ implying that $d_2 = d_3 = \ldots = d_n = 0$. Then for a group matrix A

(a) A^-_{13} exists if and only if there is an orthogonal matrix P such that $A = P \begin{bmatrix} K \\ 0 \end{bmatrix}$ where K has a right inverse.

If K_R^{-1} is a right inverse of K, then $[K_R^{-1} \quad 0]\,P^{-1}$ is a $\{1,3\}$ g-inverse of A.

(b) A^-_{14} exists if and only if there is an orthogonal matrix Q such that $A = [H \quad 0]\,Q$ where H has a left inverse. If H_L^{-1} is a left inverse of H, then $Q^{-1} \begin{bmatrix} H_L^{-1} \\ 0 \end{bmatrix}$ is a $\{1,4\}$ g-inverse of A.

The proof is contained in the proof of Theorem 1.

Proposition 1. *If A is a group matrix of order n, then A is similar to a matrix of the form* $\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, *where* A_1 *is a non-singular matrix. Then therefore* $U^t A U = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, *where U is an orthogonal matrix.*

If A_1 *is a rxr diagonal matrix with non zero diagonal elements then we write A as it is of the form. Again, if P is an idempotent matrix we can express*

$$P = U \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} U^t, \text{ where } U \text{ is orthogonal.}$$

Theorem 3 : *If A be a group matrix of rank r and U is an orthogonal matrix such that*

$$U^t A\, U = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } A \text{ is a } rxr$$

diagonal matrix with non zero diagonal elements and P is a matrix such that

$$P = U \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} U^t,$$

then, $A^+ = (A+P)^{-1} - P.$

Proof. *If* $r = 0$, *then* $A = 0$ *and* $P = I$, *then* $A^+ = 0$ *and if* $r = n$, *A is non singular and* $P = 0$, *hence* $A^+ = A^{-1}$.

For $0 < r < n$, $A + P = U \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} U^t$.

Hence,

$$(A + P)^{-1} = U \begin{bmatrix} A_1^{-1} & 0 \\ 0 & I_{n-r} \end{bmatrix} U^t,$$

and $$(A + P)^{-1} - P = U \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^t$$

$$= A^+ \text{, by theorem 1.}$$

Corollary 2. If A be an nxn group matrix and U is an orthogonal matrix, then $(A + P)^{-1}$ satisfies the Moore-Penrose equations (1),(3) and (4).

Proof. By theorem 3, $(A + P)^{-1} = A^+ + P$.

Hence, $A (A + P)^{-1} = A (A^+ + P) = AA^+$, since $AP = 0$,

therefore, $A (A + P)^{-1} A = AA^+A = A$.

Similarly, $(A + P)^{-1} A (A + P)^{-1} = A^+A (A^+ + P)$

$$= A^+ A A^+, \text{ since } AP = 0 = PA$$

and also $\{(A + P)^{-1} A\}^t = (A^+A)^t$

$$= A^+A = (A + P)^{-1}A,$$

hence proved.

CHAPTER 4.

ON GENERALIZED INVERSES OF MATRICES OVER A DIVISION RING.

Introduction. The three elements 0, 1 and 2 constitute a ring if addition and multiplication are defined by the following tables,

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

x	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

And the non zero elements 1 and 2 form a group under multiplication. Hence 0, 1 and 2 form a division ring $\mathbb{D}$ and further it is also commutative. By a matrix over $\mathbb{D}$ we mean a matrix whose elements belong to $\mathbb{D}$. We called such matrices as the Division matrices. Sums and products of such matrices are defined as in the real case. These division matrices have defferent types, some of them are idempotent, orthogonal etc. We call a matrix X over $\mathbb{D}$ (of order $n \times m$) a generalized inverse of a matrix A (of order $m \times n$) if $A X A = A$. In this chapter we study the various types of g-inverses of Division matrices. In the preliminaries we give some definitions and a few results. And after characterizing idempotent matrices, we obtain the general form of matrices which have g-inverses.

Further we define ordered division ring $\mathcal{D}_o$ and positive definite matrices over $\mathcal{D}_o$ and some results are obtained thereon.

2. Preliminaries. Let $\mathbb{D}$ be a Division ring of real numbers. By vectors and matrices over $\mathbb{D}$ we mean vectors and matrices whose elements belong to $\mathbb{D}$. Sums and products of matrices over $\mathbb{D}$ are defined as in the real case. If $x_1, \ldots, x_n$ are vectors over $\mathbb{D}$, the linear manifold $\mathcal{M}(x_1, \ldots, x_n)$ over $\mathbb{D}$ in the set of all vectors of the type $\sum_{i=1}^{n} \alpha_i x_i$ where $\alpha_i \in \mathbb{D}$. If A be a matrix over $\mathbb{D}$, $\mathcal{M}(A)$ represents the linear manifold spanned by the columns of A. We say $A \leqq B$ if $a_{ij} \leqq b_{ij}$ for all i and j.

We continue to use the notion for generalized inverse or g-inverse given in Rao and Mitra [67].

For an mxn real matrix A the unique matrix X satisfying the four equations

$$A X A = A \qquad (1)$$

$$X A X = X \qquad (2)$$

$$(AX)^t = AX \qquad (3)$$

$$(XA)^t = XA \qquad (4)$$

is called the Moore-Penrose generalized inverse of A and is denoted by A^+. A^- denotes the g-inverse of A satisfying (1), A_r^- the reflexive g-inverse of A satisfying (1) and (2), A_{13}^- the $\{1,3\}$ g-inverse of A satisfying (1) and (3), and A_{14}^- the $\{1,4\}$ g-inverse of A satisfying (1) and (4).

Proposition 1. *Let A be an mxn matrix and X be an nxm matrix. Then the following statements are equivalent.*

(a) $AXA = A$

(b) Xy is a solution of the system of linear equation $Ax = y$ whenever a solution exists i.e. $y \in \mathcal{M}(A)$.

(c) AX is idempotent and $\mathcal{M}(A) = \mathcal{M}(AX)$.

(d) XA is idempotent and $\mathcal{M}(A^t) = \mathcal{M}(A^tX^t)$.

Proof. *(a)$\Longleftrightarrow$(b), the equation $Ax = y$ is consistent if $y = AZ$ where Z is an arbitrary nx1 vector. Hence the existence of $X(=A^-)$ implies $A(XAZ) = AZ$ which gives $AXA = A$. Conversely, if the equation $Ax = y$ is consistent, then there exists a vector W such that $AW = y$. Since $AXA = A$, $AXAW = AW \Rightarrow AXy = y$; i.e. Xy is a solution of the equation $Ax = y$.*

(a)$\Rightarrow$(c), for if $AXA = A \Rightarrow AXAX = AX$, that is, if $H = AX$, $AXAX = AX \Rightarrow H^2 = H$. Hence H is idempotent and rank A = rank (AX) Rao and Mitra [67,p21]. And hence $\mathcal{M}(A) = \mathcal{M}(AX)$.

Again (a)$\Rightarrow$(d), for $AXA = A \Rightarrow XAXA = XA$. If $H = XA$, $H^2 = H$. Hence $H = XA$ is idempotent and rank A = rank $(XA) \Longrightarrow \mathcal{M}(A^t) = \mathcal{M}(A^tX^t)$.

Remark 1. *If X is a g-inverse of A denoted by A^-, then XAX is a reflexive g-inverse of A denoted by A_r^-. So the existence of A^- implies the existence of A_r^-. And if X_1 and X_2 are two possible defferent g-inverses of A, then $X_1 + X_2$ is also a g-inverse of A. If the number of nxm matrices is finite, we define a maximum g-inverse of A.*

Definition 1. *A generalized inverse X of A is said to be a maximum g-inverse of A if every* $A^{-} \leq X$.

As a consequence we see that any matrix having a g-inverse has a maximum g-inverse.

3. Generalized Inverses of Matrices Over The $\{0,1,2\}$ Division Ring.

Definition 2. *A set of vectors* $x_1, \ldots, x_n$ *is said to be independent if no vector is the sum of some of the remaining vectors and if the null vector is not in the set.*

A vector y is said to be dependent on the vectors $x_1, \ldots, x_n$ *if* $y \in \mathcal{M}(x_1, \ldots, x_n)$ *.Otherwise y is said to be independent of* $x_1, \ldots, x_n$.

It is known that a set of linearly independent vectors of a linear manifold cannot contain the null vector.

A subset S of a set T of vectors is said to be a basis (finite) of T if S is independent and $T \subset \mathcal{M}(S)$, *where* $\mathcal{M}(S)$ *is the set spanned by S.*

Every set of vectors T (which has at least one non null vector) has a basis, and it is unique, Mirsky [44,p48].

The dimension or cardinality of a basis is the number of elements of the basis. And then the rank of a set of vectors is the dimension of its basis.

The row rank R (A) or column rank C(A) of a matrix A ≠0 is the rank of the row or column vectors. The row rank and column rank of any zero matrix are both equal to zero.

Remark 2. In general the row rank and column rank of a matrix need not be equal.

For example, consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 \end{bmatrix}.$$

A is not row independent, since the sum of the first, second and third rows is equal to the fourth row. However A is column independent and R(A) = 3 and C(A) = 4.

If row rank of A = column rank of A = r, the matrix A is said to be of rank r.

Let A be a rectangular mxn matrix and suppose that rank of A is equal to m, in which case AA^* is of order mxm and rank m. The inverse $(AA^*)^{-1}$ exists and $I_m = (AA^*)(AA^*)^{-1} = A[A^*(AA^*)^{-1}] = AA_R^{-1}$, where I_m is the identity matrix of order m, that is, there exists a matrix A_R^{-1}, which postmultiplies A to I. Such a matrix is called a right inverse of A.

Similarly, if rank of $A = n$, there exists a matrix A_L^{-1} called the left inverse of A such that $A_L^{-1}A = I_n$.

One choice of A_L^{-1} is $(A^*A)^{-1}A^*$. It is clear that A_L^{-1} or A_R^{-1} exists only in special cases when rank of mxn matrix is n or m. If $n \neq m$, both A_L^{-1} and A_R^{-1} cannot exist.

Then a matrix A of order mxn is said to be non singular if row rank of A is m and the column rank is n.

An orthogonal matrix over $\mathbb{D}$ is a square matrix such that every row and every column contains exactly one 1 or one 2.

Example. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ *be a square matrix of order 3 such that every row and every column contains exactly one 1 or one 2. Since $AA^t = I = A^t A$, A is orthogonal.*

Following Rutherford [69]'s idea of two-sided inverse of a Boolean matrix, we deduce that a matrix A over $\mathbb{D}$ possesses a two-sided inverse if and only if A is an orthogonal matrix and in this case A^t is a two-sided inverse.

Thus a matrix A has an inverse if and only if it is an orthogonal matrix.

Theorem 1. *Let A be an nxn idempotent matrix. Then for some i, $a_{ii} = 0$ implies that the ith column of A depends on the rest of the columns and the ith row of A depends on the rest of the rows. Further, the matrix obtained by deleting the ith column and the ith row of A is idempotent.*

Proof. *If the generality is not lost,*

let $A = \begin{bmatrix} B & C \\ D & 0 \end{bmatrix}$, *where* $a_{nn} = 0$.

Here, we have $A^2 = A \Rightarrow \begin{bmatrix} B^2+CD & BC \\ DB & DC \end{bmatrix} = \begin{bmatrix} B & C \\ D & 0 \end{bmatrix}$.

Therefore $B^2+CD = B$, $BC = C$, $DB = D$ *and* $DC = 0$.

It implies that $\begin{bmatrix} C \\ 0 \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} C$ *and* $[D \quad 0] = D[B \quad C]$.

Again, $B^2+CD = B \Rightarrow CD \leq B \Rightarrow BCD = CD \leq B^2$.

So $B = B^2 + CD = B^2$. *Hence the theorem.*

Example. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, *where* $B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$,

then $B^2 = B$, *when third column and third row are deleted.*

Corollary 1. *If A is idempotent and non singular then* $a_{ii} = 1$ *for all i.*

Theorem 2. *Let A be an nxn idempotent matrix such that $a_{ii} = 1$ for all i. Then the ith column of A depends on the rest of the columns if and only if the ith row of A depends on the rest of the rows. Further, the matrix obtained from A by deleting the ith column and the ith row is idempotent.*

Proof. *If the generality is not lost, let the nth column of A depend on the rest of the columns, and*

let $A = \begin{bmatrix} B & C \\ D & 1 \end{bmatrix}$, *where* $a_{nn} = 1$.

Here, $A^2 = A \Rightarrow \begin{bmatrix} B^2+CD & BC+C \\ DB+D & DC+1 \end{bmatrix} = \begin{bmatrix} B & C \\ D & 1 \end{bmatrix}$.

Therefore, (i) $B^2+CD = B \Rightarrow B^2 \leq B$. *Since* $B \geq I$,

$B^2 \geq B$. *So* $B^2 = B$.

(ii) $BC + C = C \Rightarrow BC \leq C \Rightarrow BC = C$.

(iii) $DB + D = D \Rightarrow DB \leq D \Rightarrow DB = D$.

(iv) $DC = DBC$.

$\begin{bmatrix} C \\ 1 \end{bmatrix}$ *depends on the rest of the columns, then there exists a vector Z such that* $\begin{bmatrix} C \\ 1 \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} Z$.

Here, $C = BZ$, $1 = DZ$, then $DC = DBZ = DZ = 1$, which implies

$[D \quad 1] = D[B \quad C]$.

Hence the theorem is fully proved.

Corollary 2. *If A is idempotent, row rank of A = column rank of A. Hence, due to Plemmon [57]'s Theorem 1.7, A has a g-inverse and A is a g-inverse of itself [46].*

From the above two theorems a direct consequence in the form of a theorem follows.

Theorem 3. *A is an idempotent matrix of rank r if and only if there exists an orthogonal matrix U such that*

$$A = U \begin{bmatrix} B & DC \\ DB & DBC \end{bmatrix} U^t ,$$

where B is an $r \times r$ non singular idempotent matrix and C and D are such that $CD \leq B$.

From this another theorem also follows.

Theorem 4. *A is a symmetric idempotent matrix of rank r if and only if there exists an orthogonal matrix U such that*

$$A = U \begin{bmatrix} I_r & C \\ D & DC \end{bmatrix} U^t$$

where C and D are such that $CD \leq I$.

Corollary 3. *If A is non singular, idempotent and symmetric, then* $A = I$.

Proof. *Let* x_j *denote the* j*th column of* A. *Since* A *is idempotent, we have* $Ax_j = x_j$, *that is*

$$a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{ij}x_i + \cdots + a_{nj}x_n = x_j$$

Let $a_{ij} = 1$ *for some* $i \neq j$. *Then we have* $x_i \leq x_j$. *Now the fact that* A *is symmetric implies* $a_{ji} = 1$, *which also implies* $x_j \leq x_i$. *So* $x_i = x_j$, *which is a contradiction with the hypothesis that* A *is non singular.*
Therefore $a_{ij} = 0$ *for all* $i \neq j$. *From Corollary 1 it follows that* $a_{ij} = 1$ *for all* i. *So* $A = I$.

Corollary 4. *An nxn idempotent matrix* A *is given by* $A = B^{-}B$, *where* B *is an mxn matrix.*

Proof. $A^2 = B^{-}B$. $B^{-}A = B^{-}(BB^{-}B) = B^{-}B = A$. *Hence* A *is idempotent. Again* $AAA = A^2A = AA = A$, *and hence* A *is a g-inverse of itself.*

Theorem 5. *A matrix* A *has a g-inverse if and only if there exist orthogonal matrices* U *and* V *such that*

$$A = U \begin{bmatrix} B & BC \\ DB & DBC \end{bmatrix} V ,$$

where B *is non singular and idempotent and* C *and* D *are such that* $CD \leq I$.

Proof. *'If' part, let there exist orthogonal matrices* U *and* V *such that*

$$A = U \begin{bmatrix} B & BC \\ DB & DBC \end{bmatrix} V \quad .$$

Let us also consider another matrix X such that

$$X = V^t \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^t \quad .$$

Now, $$AX = U \begin{bmatrix} B & 0 \\ DB & 0 \end{bmatrix} U^t \quad ,$$

and $$AXA = U \begin{bmatrix} B & BC \\ DB & DBC \end{bmatrix} V = A \quad ,$$

and AX is idempotent. Hence A has a g-inverse X which is denoted by A^- .

'Only if' part, let A has a g-inverse X. Then AX is idempotent. Therefore from Theorem 4.3.3. there exists an orthogonal matrix U such that

$$AX = U \begin{bmatrix} B & BE \\ DB & DBE \end{bmatrix} U^t \quad ,$$

where B is an rxr non singular idempotent matrix and r is the rank of AX and D and E are such that $ED \leqq B$.

Now by the above proposition 1, $M(A) = M(AX)$.

Therefore, $A = U \begin{bmatrix} B & BC \\ DB & DBC \end{bmatrix} V$ *for some matrix C and some orthogonal matrix V.*

Remark 3. *In the above proof it can be easily seen that r is the rank of A.*

Corollary 5. *If A^- exists, then*

(i) row rank of A = column rank of A, [Plemmons, 1971]

(ii) B^- exists if $M(A) = M(B)$ or $M(A^t) = M(B^t)$.

4. M oore-Penrose Inverse And Other Types Of g-inverses.

We discuss the necessary and sufficient conditions for a matrix A to possess various types of g-inverses viz, A_{13}^-, A_{14}^- and A^+.

Theorem 6. *For a matrix A the following are equivalent.*

(a) A_{13}^- exists.

(b) A is of the form $U\begin{bmatrix} I & B \\ C & CB \end{bmatrix} V$, where U and V are orthogonal matrices and C satisfies $C^tC \leq I$, in this case

$$V^t \begin{bmatrix} I & C^t \\ 0 & 0 \end{bmatrix} U^t \text{ is } A_{13}^- .$$

(c) A^- exists and $M(A^t) = M(A^tA)$.

(d) There exists a matrix X such that $A^tAX = A^t$.

Proof. *(a) $\Rightarrow$ (b): let $X = A_{13}^-$. Then AX is symmetric and idempotent. Therefore there exists an orthogonal matrix U such that*

$$AX = U\begin{bmatrix} I & C^t \\ C & CC^t \end{bmatrix} U^t, \text{ where } C \text{ is such that } C^tC \leq I$$

(by Theorem 4).

Again, X is A^- implies that $M(A) = M(AX)$, which implies that A is of the form $U\begin{bmatrix} I & B \\ C & CB \end{bmatrix} V$ for some C and some orthogonal U.

(b) $\Rightarrow$ (c); the assumption of (b) implies the existence of A^-, which again implies that

$$M(A^t) = M((XA)^t) = M(A^tA).$$

(c) $\Rightarrow$ (d), the existence of A^- and $M(A^t) = M(A^tA)$ implies that A^tA exists and there exists a matrix D such that $A = DA^tA$.

Take $X = (A^tA)^-A^t$.

Now, $A^tAX = A^tA(A^tA)^-A^t = A^tA(A^tA)^-A^tAD^t = A^tAD^t = A^t$.

(d) $\Rightarrow$ (a); that $A^tAX = A^t$ implies that AX is symmetric and hence $AXA = A$. So X is A^-_{13}.

The next two theorems follow from the above theorem.

Theorem 7. *For a matrix A, the following are equivalent.*

(a) A^-_{14} exists.

(b) A is of the form $U\begin{bmatrix} I & B \\ C & CB \end{bmatrix} V$, where U and V are orthogonal matrices and B satisfy $BB^t \leq I$, in which case

$$V^t \begin{bmatrix} I & 0 \\ B^t & 0 \end{bmatrix} U^t \text{ is an } A_{14}^- .$$

(a) A^- exists, and $M(A) = M(AA^t)$.

(d) There exists an X such that $XAA^t = A^t$.

Theorem 8. *For a matrix A the following are equivalent.*

(a) A^+ exists.

(b) A is of the form $U \begin{bmatrix} I & B \\ C & CB \end{bmatrix} V$, where U and V are orthogonal matrices, and B and C are such that $BB^t \leq I$ and $C^tC \leq I$, in which case $A^t = A^+$.

(c) A^- exists, $M(A) = M(AA^t)$ and $M(A^t) = M(A^tA)$.

(d) There exists a matrix X such that $XAA^t = A^t$ and $A^tAX = A^t$.

5. New Dimension To Division Ring And Positive Definite Matrices Over It.

Definition 3. *Ordered Division Ring.*

A field F is ordered if there is a non empty subset P, the elements of which are referred to as positive such that P is closed with respect to addition and multiplication, and such that every elements a of F satisfies just one of the following conditions,
either a is positive or a is zero or -a is positive (a is negative) [45,p69].

From the definition of an ordered field, we deduce that the $\{0,1,2\}$ division ring $\mathbb{D}$ formed under the operation of addition and multiplicatio- n defined in the introduction of this chapter in ordered division ring denoted by $\mathcal{D}_o$.

Definition 4. Positive Definite Matrices.

A symmetric nxn matrix A with elements in an ordered field F is positive definite (p.d.) if its rank and index are both n. If F is the field of reals, then A is p.d. if and only if $A \tilde{C} I_n$ (i.e. A is congruent to I_n, $A = U^t I_n U$).

Thus A is p.d. if and only if there is a real, unit matrix U such that $A = U^tU$ [45,p57 & 71].

In the same fashion, a symmetric nxn matrix A upon $\mathcal{D}_o$ is p.d. if and only if there is a real, unit matrix U such that $A = U^tU$.

1. Factorization Of Hermitian Positive Definite Matrices.

Let A be an nxn symmetric p.d. matrix over $\mathcal{D}_o$. Then we may write

$A = U^tDU$, where U is orthogonal and $D = \text{diag } \{d_1, \ldots, d_n\}$, where $d_i \neq 0$.

Result 1. If D is a diagonal matrix with diagonal elements equal to 0 and 1, it is complemented that A is a g-inverse of itself [67,p 42].

Result 2. Given a matrix A of mxn order, there exist non singular matrices B and C of orders m and n respectively, such that $BAC = D$,

where D is a diagonal matrix, with non zero elements from $\mathbb{D}$ (the $\{0,1,2\}$division ring) in the main diagonal.

Then, $A^- = CD^-B$ is a reflexive g-inverse of A [67,p 209].

If D^- is a matrix obtained from $\mathbb{D}$ by taking the transpose and replacing the non zero elements by their reciprocals, it is seen that D^- is also D^+ [67, p 209].

Result 3. Since 1 and 2 are the reciprocals of 1 and 2, $D^- = D^+$ and also $A^- = CD^-B = CD^+B = A^+$ if B and C are orthogonal [67, p209].

CHAPTER 5

HOMOMORPHISM OF MATRICES OVER AN ARBITRARY DIVISION RING.

Introduction. Herstein [34] has given the definition of homomorphism as follows.

A mapping ϕ from a ring R into another ring R' is said to be a homomorphism if

$$(1) \quad \phi(a+b) = \phi(a) + \phi(b)$$

$$(2) \quad \phi(a.b) = \phi(a).\ \phi(b)$$

for all $a,b \in R$.

The + and . occuring on the left-hand sides of the relations (1) and (2) are those of R whereas the + and . occuring on the right- hand sides are those of R'.

In this chapter we define a homomorphism on matrices over a finite division ring and some results in the form of theorems are obtained such as, for a homomorphism f and some matrices A and B over a division ring, $f(A) = f(B)$ if and only if $A = B$ and X is a g-inverse of A if and only if $f(X)$ is a g-inverse of $f(A)$ etc. We define orthogonal and partial orthogonal matrices over a finite division ring and some results are given in the form of theorems. We extend few results of the theorems on any arbitrary division ring. In section 3 of this chapter the idea

of anti-isomorphism of Jacobson [37] has been used to the semi-groups of generalized inverses of matrices over any arbitrary division ring.

2. Preliminaries And Notations.

In chapter 4 we have shown that the three elements 0,1 and 2 form a division ring if addition and multiplication are defined by the following rules,

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

x	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

.

This division ring is denoted by $\mathbb{D}_0$ in this chapter.

Definition 1. An orthogonal matrix over $\mathbb{D}_0$ is a square matrix such that every row and every column contains exactly one 1 or one 2, e.g. a 3x3 orthogonal matrix over $\mathbb{D}_0$ can be shown by $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Cline [16] defined *- orthogonal matrices as matrices U and V of the same size with both $UV^* = 0$ and $U^*V = 0$, where 0 is the null matrix and * is the conjugate transpose.

Definition 2.

Let $\mathbb{D}_0$ denote the $\{0,1,2\}$ division ring and $\mathbb{D}$ a finite division ring of real numbers.

A mapping $f : \mathbb{D} \to \mathbb{D}_0$ is called a homomorphism if

$$(1) \quad f(a+b) = f(a) + f(b)$$

$$(2) \quad f(a.b) = f(a) \,.\, f(b)$$

If $a_{ij} \in \mathbb{D}$ and $f(a_{ij}) \in \mathbb{D}_0$, then for a homomorphism f and a matrix A over $\mathbb{D}$, we define a matrix $f(A)$ over $\mathbb{D}_0$ by $(f(A))_{ij} = f(a_{ij})$, where $f(a_{ij})$ is the element of ith row and jth column of the matrix $f(A)$, which is also denoted by $(f(A))_{ij}$. Then it is easy to see that

$$(i) \quad f(A + B) = f(A) + f(B)$$

$$(ii) \quad f(A.B) = f(A).f(B)$$

$$(iii) \quad f(A^t) = f(A)^t$$

where A and B are matrices on $\mathbb{D}$ and t stands for transpose of a matrix. Let F denote the set of all homomorphisms from $\mathbb{D}$ to $\mathbb{D}_0$.

The following two lemmas are established.

Lemma 1.

For matrices A and B over $\mathbb{D}$, and for all $f \in F$, $f(A) = f(B)$ if and only if $A = B$.

This lemma is a direct consequence of the fact that if $a \neq b$ then there exists $f \in F$ such that $f(a) \neq f(b)$.

Similarly $f(a) \neq f(b)$ implies $a \neq b$ if the inverse function f^{-1} exists.

By the above lemma, we find that X is a g-inverse of A if and only if f(X) is a g-inverse of f(A) for all $f \in F$.

For, if X is a g-inverse of A then $AXA = A \Longrightarrow f(AXA) = f(A)$ or $f(A)\,f(X)\,f(A) = f(A)$. Hence f(X) is a g-inverse of f(A).

Lemma 2.

Given labeled matrices (or indexed matrices) A_f over $\mathbb{D}_0$ indexed by $f \in F$ there exists a matrix A over $\mathbb{D}$ such that $f(A) = A_f$ for all $f \in F$. Then this A is unique.

Before starting the proof of this lemma we recall the Boolean Algebra consideration of Elliot Mendelson [20,p.62] which is given below.

A function ϕ is called an isomorphism from a Boolean algebra

$$\mathcal{B} = \langle B, \wedge_{\mathcal{B}}, \vee_{\mathcal{B}}, '_{\mathcal{B}}, 0_{\mathcal{B}}, 1_{\mathcal{B}} \rangle$$

into a Boolean algebra $\mathcal{C} = \langle C, \wedge_{\mathcal{C}}, \vee_{\mathcal{C}}, '_{\mathcal{C}}, 0_{\mathcal{C}}, 1_{\mathcal{C}} \rangle$ if and only if

(a) ϕ is one-one function from B into C,

(b) for any x, y in B,

$$\phi(x \wedge_{\mathcal{B}} y) = \phi(x) \wedge_{\mathcal{C}} \phi(y)$$

$$\phi(x \vee_{\mathcal{B}} y) = \phi(x) \vee_{\mathcal{C}} \phi(y)$$

$$\phi(x'_{\mathcal{B}}) = (\phi(x))'_{\mathcal{C}}.$$

Such a function ϕ is called an isomorphism from $\mathcal{B}$ onto C if, in addition, ϕ is a function from B onto $\mathcal{C}$.

The theorem 3.13 of [20] states that,

Let ϕ be an isomorphism from a Boolean algebra $\mathcal{B}$ into (respectively, onto) a Boolean algebra C. Then

(a) $\phi(0_{\mathcal{B}}) = 0_C$ and $\phi(1_{\mathcal{B}}) = 1_C$ etc.

It is also stated that, the inverse ϕ^{-1} is the function whose domain is the range $\phi[B]$ of ϕ (here, $\phi[B] = \{\phi(x) | x \in B\}$) and such that for any y in $\phi[B]$, $(\phi^{-1})(y)$ is the unique x in B such that $\phi(x) = y$.

Proof of Lemma 2.

The proof follows if it can be proved that given elements $a_f \in \mathbb{D}_o$ indexed by $f \in F$, there exists a unique element $a \in \mathbb{D}$ such that $f(a) = a_f$ for all $f \in F$.

From Elliot Mendelson's Boolean algebra consideration, let the inverse function f^{-1} exist. Then f^{-1} is a function whose domain is the range $f[\mathbb{D}]$ of f(here $f[\mathbb{D}] = \{f(a) | a \in \mathbb{D}\}$) and such that, for any a_f in $f[\mathbb{D}]$ ($= \mathbb{D}_o$), $f^{-1}(a_f)$ is the unique a in $\mathbb{D}$ such that $f(a) = a_f$.

Definition 3.

A square matrix over $\mathbb{D}$ is called an orthogonal matrix if

(i) elements of any row are pairwise disjoint i.e. a.b = 0, where a and b are different elements of the same row;

(ii) elements of any column are pairwise disjoint;

(iii) the sum of all the elements in any row is equal to 1 or 2 ;
(iv) the sum of all the elements in any column is equal to 1 or 2.

A matrix over $\mathbb{D}$ is called a partial orthogonal matrix if it satisfies conditions (i) and (ii) above.

Our terminology are justified in view of the following :

Lemma 3.

(a) $\mathcal{O}$ is an orthogonal matrix over $\mathbb{D}$ if and only if $f(\mathcal{O})$ is an orthogonal matrix over $\mathbb{D}$ for all $f \in F$.

(b) $\mathcal{O}$ is a partial orthogonal matrix over $\mathbb{D}$ if and only if $f(\mathcal{O})$ is a partial orthogonal matrix over $\mathbb{D}_0$ for all $f \in F$.

Before starting the proof of this lemma we recall the lemma 3.3 of Herstein [34,p92] which states that if ϕ is a homomorphism of a ring R into another ring R' then

(1) $\phi(0) = 0$, (2) $\phi(-a) = -\phi(a)$ for every $a \in R$.

A word of caution : If both R and R have the respective unit elements 1 and $1'$ for their multiplications it need not follow that $\phi(1) = 1'$. However, if R' is an integral domain, or if R' is arbitrary but ϕ is onto, then $\phi(1) = 1'$ is indeed true.

Proof of Lemma 3.

(a) Let $\mathcal{O}$ be orthogonal, then $a.b = 0$, where $a \neq b$ are elements of the same row and $f(a)$. $f(b) = f(ab) = f(0) = 0$. It shows that $f(a) \neq f(b)$ are pairwise disjoint elements of the same row of $f(\mathcal{O})$. Similarly the elements of any column are also pairwise disjoint. Further if $a+b = 1$ or 2, then $f(a) + f(b) = f(a + b) = f(1)$ or $f(2) = 1$ or 2. It also shows—

that the sum of the elements in any row or column is equal to 1 or 2. The converse is trivial.

(b) The proof is the same as above.

We prove the following extensions.

Theorem 1.

For square matrices A and B over $\mathbb{D}$ if $AB = I$, then A is an orthogonal matrix over $\mathbb{D}$ and $B = A^t$.

Proof.

Let $AB = I$, then $f(AB) = f(I)$ for all $f \in F$, or $f(A)\,f(B) = I$. It implies that $f(A)$ is an orthogonal matrix over $\mathbb{D}_0$, and $f(B) = f(A)^t = f(A^t)$. Therefore it implies that A is an orthogonal matrix over $\mathbb{D}$ and $B = A^t$.

Theorem 2.

If a matrix A over $\mathbb{D}$ has a g-inverse, then it has a partial orthogonal matrix over $\mathbb{D}$ as a g-inverse.

Proof.

Suppose A has a g-inverse, then $f(A)$ has a g-inverse for all $f \in F$. It also implies that $f(A)$ has a partial orthogonal g-inverse, say X_f over $\mathbb{D}_0$ indexed by f. Then also there is a partial orthogonal matrix X over $\mathbb{D}$ such that $f(X) = X_f$, and this X is a g-inverse of A.

Theorem 3.

If A^+ exists, then it is A^t.

Proof.

Let X be A^+. Then

$f(X) = (f(A))^+$ for all $f \in F$.

It implies that $f(X) = (f(A))^t = f(A^t)$ for all $f \in F$. Therefore it implies that $X = A^t$.

Remark 1.

All the above results can be extended to matrices over any arbitrary division ring $\mathbb{D}$. The reason is that, if one wishes to prove a result about a matrix A over $\mathbb{D}$, it is enough to consider A as a matrix over a finite division ring generated by the elements of A, in view of the following lemma.

Lemma 4.

Let A be a matrix over an arbitrary division ring $\mathbb{D}$. It has a g-inverse over $\mathbb{D}$ if and only if it has a g-inverse over the division ring generated by the elements of A.

Proof.

Let A be a matrix over an arbitrary division ring $\mathbb{D}$, and let $M(A)$ be also a division ring generated by the columns of A. Let F be a set of homomorphisms from $\mathbb{D}$ to $M(A)$.

We recall lemma 2 (replacing $\mathbb{D}_0$ by $M(A)$) which states that, if A_f are given matrices over $M(A)$, indexed by $f \in F$, there exists a matrix A over $\mathbb{D}$ such that $f(A) = A_f$, for all $f \in F$. This matrix A is unique.

Using this lemma 2 and writting $f(A) = A_f$, we have that the matrix A over the arbitrary division ring $\mathbb{D}$ is unique.

Let A_f has a g-inverse X_f over $M(A)$ such that

$$A_f X_f A_f = A_f$$
$$\text{or } f(A)\, f(X)\, f(A) = f(A)$$
$$\text{or } f(AXA) = f(A).$$

It implies that $A X A = A$. Hence X is a g-inverse of A over $\mathbb{D}$

.

The converse is trivial.

3. Anti - Isomorphism And Its Application To Semi-Groups Of Generalized Inverses Of Matrices.

The idea of Jacobson [37,p71]'s anti-isomorphism is used here. Before giving the definition of anti-isomorphism, we deal with some terms used in it.

1. Quaternions.

Let Q be the set of matrices in C_2, C the field of complex numbers, that have the form

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} a_0 + a_1\sqrt{-1} & a_2 + a_3\sqrt{-1} \\ -a_2 + a_3\sqrt{-1} & a_0 + a_1\sqrt{-1} \end{bmatrix} \quad a_i \text{ real} .$$

Then Q , $+$, $.$ is an instance of a subring of the ring C_2 , $+$, $.$, and ultimately Q is a division ring. And Q is also called the ring of quaternions and the elements of Q quaternions.

Further, $i = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}$, $j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $k = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}$

are quaternions. It is verified that

$$a = \begin{bmatrix} a_0 + a_1\sqrt{-1} & a_2 + a_3\sqrt{-1} \\ -a_2 + a_3\sqrt{-1} & a_0 - a_1\sqrt{-1} \end{bmatrix} = a_0' + a_1' i + a_2' j + a_3' k .$$

Jacobson defined anti-isomorphism as a mapping from a ring R onto another ring $\bar{R}$ of conjugates given by a to $\bar{a}$ where $a = a_0 + a_1 i + a_2 j + a_3 k$ is a quaternion and $\bar{a} = a_0 - a_1 i - a_2 j - a_3 k$ is the conjugate of a, and this mapping is one-one that satisfies the relations

$$\overline{a + b} = \bar{a} + \bar{b} ,$$

and

$$\overline{ab} = \bar{b}\bar{a} .$$

Our homomorphism $f : \mathbb{D} \longrightarrow \mathbb{D}_0$ is one-one and let it satisfy

$$f(a + b) = f(a) + f(b) ,$$

and $f(ab) = f(b)\, f(a)$,

then we can have f as anti-isomorphism.

2 Application Of Anti-Isomorphism To Semi-Groups Of g-Inverses Of Matrices.

Let S be a set of all matrices having g-inverses over a division ring and S^+ the set of the g-inverses. Evidently S is a semi-group and for all matrices $A, B \in S$ there exist unique A^+, B^+ in S^+.

Many mathematicians like Greville [24], Searle [75], Shinozaki [74], Rao and Mitra [67] etc. have given necessary and sufficient conditions for the Moore-Penrose inverse of a complex matrix product AB to be expressed as $(AB)^+ = B^+A^+$. Rao and Mitra [67,p67,5 complements] have stated that if A is $m \times r$ of rank r and B is $r \times n$ of rank r, then $(AB)^+ = B^+A^+$.

Therefore we say that for all A^+, $B^+ \in S^+$, there exists $A^+B^+ = (BA)^+$ in S^+, hence S^+ is a semi-group.

We define a mapping from S to S^+ such that A^+ is the map of A under this mapping. Clearly it is one-one mapping and Cline [16] has shown that $(A+B)^+ = A^+ + B^+$ if A and B are *- orthogonal. And also from above paragraph, we have $(AB)^+ = B^+A^+$. Hence this mapping from S onto S^+ satisfying all the three necessary conditions, is an anti-isomorphism.

CHAPTER 6.

SOME APPLICATIONS OF GENERALIZED INVERSES OF MATRICES TO A NEW DIMENSION OF A GROUP+.

1. Introduction.

It is well-known that a ring R is regular if for every $a \in R$, $a \in aRa$ and unit regular if for every $a \in R$, there is a unit $u \in R$ such that $a\,u\,a = a$ [21]. Hartwig and Luh [32] stated that any solution $\bar{a}$ to $axa = a$ is called an inner or 1-inverse of a, while any solution a^* to $axa = a$ and $xax = x$ is called a reflexive or 1-2 inverse of a.

[32] stated that a ring element $a \in R$(with unity) is unit regular when there is a unit $u \in R$ and a group G in R such that $a \in uG$.

[32] also proved a theorem 2B, which stated as follows :

If R is a ring with unity 1 and if $a \in R$ is a regular element, then the following are equivalent to a being unit regular.

(7) $a \in u^{-1}\,\bar{a}\,aRa\,\bar{\bar{a}}\,u^{-1}$ for some unit u and some inner inverses $\bar{a}, \bar{\bar{a}}$ in R,

(8) $\bar{a}\,x\,a = y$, $a\,y\,\bar{\bar{a}} = x$, where $\bar{a}$, $\bar{\bar{a}}$ are inner inverses of $a \Rightarrow x \approx y$ i.e. x is similar to y.

Pseudo- similarity for matrices over a field was introduced by Hartwig and Hall [31]. It was defined as follows :

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Let R_{mxn} denote the mxn matrices over a ring with unity. If $A \in R_{mxm}$ and $B \in R_{nxn}$, A is said to be pseudo-similar to B, via X and it is written $A \underset{\sim}{\sim} B$, if there exists $X \in R_{mxn}$ and two possible distinct X^{-} and $\bar{\bar{X}} \in R_{nxm}$ such that

$$X^{-}AX = B \qquad (1)$$

$$XB\bar{\bar{X}} = A \qquad (2)$$

$$XX^{-}X = X \qquad (3)$$

$$X\bar{X}X = X. \qquad (4)$$

Any solution to $XMX = X$ will be denoted by X^{-}, $X^{\bar{}}$ etc, while X^{+} denotes any solution to $XMX = X$ and $MXM = M$. These generalized inverses are sometimes called 'inner' and 'reflexive' inverses of X respectively.

In general pseudo-similarity does not imply similarity.

The authors [31] stated some properties of $\underset{\sim}{\sim}$ in the form of theorems, and the theorem 2 stated the following :

Let $A, B \in F_{nxn}$(i.e. A and B are nxn matrices over the field F). Then $A \underset{\sim}{\sim} B \Leftrightarrow A \approx B$.

Lastly the authors closed with the conjecture that the theorem 2 holds for matrices over a unit-regular ring, for which each $a \in R$ has at least one unit (= invertible) inner inverse $\bar{a}$.

Let $\mathbb{D}$ denote a division ring and $\mathbb{D}_{nxn}$ the set of all nxn matrices over $\mathbb{D}$. Then $\mathbb{D}_{nxn}$ is unit regular, for if $A \in \mathbb{D}_{nxn}$ there is

a unit $U \in \mathbb{D}_{nxn}$ such that $AUA = A$.

Example. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $U = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ be 2x2 matrices over $\mathbb{D}$.

Then $AUA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$.

Hence the pseudo-similarity of matrices implies similarity over a division ring [31] and [33]. In symbol we write, if $A,B \in \mathbb{D}_{nxn}$, then $A \approx B \Leftrightarrow A \simeq B$.

In this chapter we study mainly that similar matrices(Particularly singular++) of the same order over a division ring $\mathbb{D}$ form a semi-group with identity. And the generalized inverses of these singular matrices also form a semigroup. By using sections 2,3 and 4 of Greville [23], we attempted that rectangular or singular matrices form a new dimension to the group, called 'Matroid' with idempotent matrix as identity and few properties are also discussed thereon.

2 Main Results.

1. Since the associative law of multiplication holds in $\mathbb{D}_{nxn}$, $\mathbb{D}_{nxn}$ is a semigroup with identity I_n. Therefore singular similar matrices of the same order over $\mathbb{D}$ form a semigroup.

We show that g-inverses of singular matrices over $\mathbb{D}$ form a semi

++ Singular matrices form a group called Matrix group, Mirsky [44,p22]

group.

Here all the matrices are singular, similar and are of the same order.

Let the matrices over $\mathbb{D}$ have their g-inverses. And let G^+ denote the collection of all the g-inverses of the nxn matrices over $\mathbb{D}$.

Here if $A,B \in \mathbb{D}_{nxn}$ then $A^+, B^+ \in G^+$. And since $AB \in \mathbb{D}_{nxn}$, then $(AB)^+$ also exists in G^+.

Many authors like Greville [24] , Searle [75] , Rao and Mitra [67], Shinozaki [74] , etc have given necessary and sufficient conditions for the Moore-Penrose inverse of a complex matrix product AB to be expressed as

$$(AB)^+ = B^+A^+ .$$

Rao and Mitra [67,p67,5 complements] have stated that, if A is mxr of rank r and B is rxn of rank r, then

$$(AB)^+ = B^+ A^+ .$$

Then we deduce that if A and B each of rank $r < n$ are in $\mathbb{D}_{nxn}$, then $(AB)^+ = B^+ A^+$.

Thus, for every $A^+ , B^+ \in G^+$, there exists $(AB)^+$ in G^+ such that $B^+ A^+ = (AB)^+$ in G^+ .

The associative law also holds in G^+ , for

$$(A^+B^+)C^+ = A^+(B^+C^+) .$$

Hence G^+ is a semi-group.

Let G be the collection of all nxn matrices and their g-inverses over $\mathbb{D}$. Evidently G is a semi-group.

2. The g-inverse defined by Moore [42] and Penrose [56] was termed as pseudoinverse by Drazin [19] and Greville [23] used pseudoinverse in his papers.

We use sections 2 , 3 and 4 of Greville [23].

Section 2 stated that , if B be an nxm matrix of rank m while the columns of B are linearly independent , then the pseudoinverse of B was given as

$$B^{+} = (B^{t}B)^{-1}B^{t} \qquad (5)$$

where t stands for the transpose. For $m < n$, the pseudoinverse is a left inverse of B, which is the only left inverse of B having rows in the row space of B^{t}.

Similarly, the pseudoinverse of an mxn matrix C of rank m was given by

$$C^{+} = C^{t}(CC^{t})^{-1}. \qquad (6)$$

This is the only right inverse of C having columns in the column-space of C^{t}.

In section 3 the author [23] considered the general case of any matrix A whose rank r may be less than its smaller dimension m. Let B denote an nxr matrix whose r columns form a basis for the column-space of A. Similarly let C denote an r-rowed matrix whose rows form a basis for the row-space of A. The pseudoinverses of B and C are then given

by formulas (5) and (6).

The author stated that A has a unique left identity matrix with rows in the row-space of A^t and that was shown to be

$$I_L = BB^+. \qquad (7)$$

Evidently $I_L B = B$, and $I_L A = A$ (8)

follows, since each column of A is a linear combination of the columns of B.

Similarly,

$$I_R = C^+C \qquad (9)$$

is the only matrix with columns in the column-space of A^t which satisfies the relation

$$AI_R = A. \qquad (10)$$

It was shown that I_L and I_R are both symmetric and idempotent.

In section 4 it was shown that for any matrix A, there is a unique matrix A^+, which has its rows and columns in the row-space and column-space of A^t and also satisfies the equations

$$AA^+ = I_L, \; A^+A = I_R. \qquad (11)$$

This matrix A^+ was called the pseudoinverse of A.

3. Definition of Metroid :

Let M denote a set of all rectangular or singular matrices that possess all the properties in sections 2,3 and 4 of Greville [23]. Then for an nxm matrix B of rank m in M there is a unique B^+ (called pseudoinverse of B) given in equation (5). This B^+ is the left inverse of B such that

$I_L = BB^+$, where I_L is the unique left identity matrix defined in equation (7). Similarly for an mxn matrix C of rank m in M there is a right inverse C^+ in M such that $I_R = C^+C$, where I_R is the right identity matrix. Evidently $I_L B = B$ and $CI_R = C$. These uniques I_L and I_R are symmetric and idempotent and are not always equal.

Ultimately, we define that such a set M is called a 'Matroid' under the operation of multiplication with I_L or I_R as the identity matrix.

The associative law for multiplication holds in M, hence we called M associative matroid.

The order of a finite matroid M is defined to be the number of matrices (or elements) in M and is denoted by $|M|$; M is infinite if $|M|$ is infinite and finite if $|M|$ is finite.

Definition 1. *Left submatroid and right submatroid.*

Mitra, S.K.[48, p110] stated that an idempotent matrix is a reflexive g-inverse of itself. We deduce that if E be a symmetric idempotent matrix then E is a unique g-inverse (or pseudoinverse) of itself satisfying all the four equations of Penrose [56].

Let I_L be a symmetric idempotent matrix in M and let $M(I_L)$ denote the set of all matrices A in M such that $I_L A = A$ and $AB = I_L$ for some $B \in M$. Then $M(I_L)$ is called a left submatroid of M with the left unique identity I_L.

Similarly we define a right submatroid $M(I_R)$ of a matroid M, as the set of all matrices A in M such that $AI_R = A$ and $BA = I_R$ for some B in M.

Furthermore, if M_1 is any left submetroid of M containing I_L then clearly $M_1 \subseteq M(I_L)$ and then $M(I_L)$ is called the maximal left submetroid of M associated with I_L. Similarly we define a maximal right submetroid of M associated with I_R.

4. Some Properties Of Metroid M.

Property 1.

The metroid M is simple.

Proof.

Let $A \in M$ and $B \in M(I_L)$, then $ABA^* \notin M(I_L)$, since $AB \neq BA$ in general and $I_L A^* \neq A^*$.

And $A^* I_L A = A^* A \neq I_L \in M(I_L)$. Hence there is no composition series in M.

Proposition 1.

For any matrix A, we define index A or ind A as the smallest non-negative integral k such that rank A^k = rank A^{k+1}.

Following Drazin [19] we say that an element A of M is pseudoinvertible in M if and only if there is X in M such that $XAX = X$, $AX = XA$ and $A^{k+1} X = A^k$. If such X exists, it is uniquely determined and is called the pseudoinverse of A and is denoted by $A^{(d)}$. The pseudoinverses of singular matrices were also discussed in chapter 3.

Property 2.

A non zero idempotent I_L of the metroid M is said to be primitive if there is no nonzero idempotent other than I_L itself, satisfying the

equation $l_L A = A$. Similarly for l_R also.

A simple matroid M containing a primitive idempotent is called completely simple. Munn [48] stated in Theorem 2, that a simple semi-group is completely simple if and only if it is pseudoinvertible.

Since the matroid M is also pseudoinvertible it is costumarily completely simple.

Property 3.

Munn [47] stated that a semi-group S is said to be regular if for any given element $a \in S$ there is at least one solution of the equation $axa = a$. Hence in a regular semi-group the equations $axa = a$, $xax = x$ have at least one common solution. We conclude that the matroid M is regular, for the equations $AXA = A$ and $XAX = X$ have at least one common solution A^+ in M.

Property 4.

Since every element in M is of infinite order, the matroid M is torsion-free.

REFERENCES

1. Anderson, C.L. (1967) : A geometric theory of pseudoinverses and some applications in Statistics. Master's thesis, Southern Methodist University.

2. Albert, A. and R.W.Sittler (1965) : A method of computing least squares estimators that keep up with the data, SIAM J. Control, Ser A : 3,(384-417).

3. Aitchison, P.W. (1982) : Generalized inverse matrices and their applications. Internat. J. Math. Ed. Sci. Tech. 13. No. 1, (99-109).

4. Ben-Israel, A. and T.N.E.Greville (1974) : Generalized Inverses, Theory and Applications, Wiley Interscience, New-York.

5. Ben-Israel, A. (1969) : On matrices of index zero or one, SIAM J. Appl. Math. 17, No.6. (1118-1121).

6. Ben-Israel, A. (1965a) : A modified Newton-Raphson method for the solution of systems of equations. Israel J. Math., 3, (94-98).

7. Ben-Israel, A. and A. Charnes (1968a) : An explicit solution of a special class of linear programming problems. Operation Research, 16, (1166-1175).

8. Ben-Israel, A. and A. Charnes (1963b) : Generalized inverses and the Bott- Duffin network analysis, J. Math. Anal. Appl. 7, (428-435).

9. Batigne, D. (1978) : Integral Generalized Inverses of integral matrices, Linear Algebra and its Applications, 22, (125-134).

10. Batigne, D.R., F.J. Hall and I.J. Katz (1978) : Further results on integral generalized inverses of integral matrices, Linear and Multilinear Algebra, 6, (233-241).

11. Bell, C-Leary. (1981) : Generalized inverses of circulants and generalized circulant matrices, Linear Algebra and its Applications, 39, (133-142).

12. Bhimasankaran, P.; Majumdar, Dibyen (1980) : Hermitian and non negative definite solutions of some matrix equations connected with distribution of quadratic forms. Sankhyā Ser. A 42, no. 3-4, (272-282).

13. Bott, R. and R.J. Duffin (1953) : On the algebra of networks, Trans. Amer. Soc., 74, (99-109).

14. Barreto, G. and Jose, A. (1976) : Applications of linear Algebra, Bol. Math. 10, no. 1-6, (61-77).
(linear approximation by means of least squares)

15. Cline, R.E. (1964) : Representations for the generalized inverse of a partitioned matrix, J. Soc. Indust. Appl. Math. 12, No. 3, (588-601).

16. Cline, R.E. (1965) : Representations for the generalized inverse of sums of matrices, J. SIAM. Numer. Anal. Ser. B. Vol.2. No.1, (99-114).

17. Cline, R.E. (1963) : Representations for the generalized inverse of matrices with applications in linear programming. Doctoral dissertation, Purdue University.

18. Chien-Fu Wu (1980) : On some ordering properties of the generalized inverses of non negative definite matrices, Linear Algebra and its Applications, 32, (49-60).

19. Drazin, M.P.(1958) : Pseudoinverses in associative rings and semigroups, Amer. Math. Monthly,65, (506-514).

20. Elliot Mendelson (1970) : Theory and problems of Boolean Algebra and Switching circuits, Mc graw-Hill Book company, New-York.

21. Ehrlich, G. (1968) : Unit regular rings, Portugal Math, 27, (209-212).

22. Greville, T.N.E. (1965) : Some applications of the pseudoinverse of a matrix, SIAM Review, 2. No,1, (15-22).

23. Greville, T.N.E. (1959) : The pseudoinverse of a rectangular or singular matrix and its application to the solution of system of linear equations, SIAM Rev. , 1, No. 1, (38-43).

24. Greville, T.N.E. (1966) : Note on the generalized inverse of a matrix product, SIAM Rev. 8, (518-521).

25. Greville, T.N.E. (1957) : The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations, SIAM News Letter, 5, (3-6).

26. Graybil, F.A. (1969) : Introduction to matrices with Application in statistics. Wadsworth, Belmont, calif.

27. Golub, G. (1965) : Numerical methods for solving least squares problems, Numer. Math. , 7, (206-216).

28. Golub, G (1968) : Least squares, singular values and matrix approximations, Appl. Math. , 13, (44-51),

29. Hall, F.J. and I.J.Katz : (1980) More on integral generalized inverses of integral matrices, Linear and multilinear Algebra, vol. 9, (201-209).

30. Hall, F.J. and I.J.Katj : (1981) Non negative integral generalized inverses, Linear Algebra and its Applications, 39, (23-39).

31. Hartwig, R.E. and F.J. Hall (1978) : Pseudo-Similarity for matrices over a field, Pro. Amer. Math. Soc. vol 71, No.1, (6-10).

32. Hartwig, R.E. and J.Luh (1977) : A note on the group structure of unit regular ring elements, Pacific Journal of Mathematics, vol 71, No.2, (449-461).

33. Hartwig, R.E. and Mohan S. putcha (1981) : Semisimilarity for matrices over a division ring, Linear Algebra and its applications, 39, (125-132).

34. Herstein,I. (1975) : Topics in Algebra, Vikas publishing House pvt. Ltd. New Delhi.

35. Hestenes,M.R. (1961) : Relative hermitian matrices, Pacific J. Math. 11, (225-245).

36. Hall,F.J. ; Meyer,Carl D, Jr. (1975) : Generalized inverses of the fundamental bordered matrix used in linear estimation. Sankhyā Ser.A 37. No.3, (428-438).

37. Jacobson, N. (1951) : Lectures in Abstract Algebra, vol. I, Van Nostrand.

38. Jacobson, N. (1953) : Lectures in Abstract Algebra, vol. II, van Nostrand.

39. Kenneth Hoffman and Ray Kulje (1972) : Linear Algebra, Prentice Hall of India Pvt. Ltd. New Delhi.

40. Khatri, C.G. and C.R.Rao (1968) : Solution to some functional equations and their applications to characterisation of probability distribution. Sankhyā, Ser. A, 30, (167-180).

41. Lewis, T.O. (1967b) : A theory of linear estimation, Texas center for Research.

42. Moore, E.H. (1920) : On the reciprocal of the general algebraic matrix (abstract), Bull. Amer. Math. Soc. 26, (394-395).

43. Moore, E.H. (1935) : General Analysis, American Philosophical Society, Philadelphia.

44. Mirsky, L. (1955) : An introduction to Linear Algebra, Oxford University Press, Oxford.

45. Morris Newman (1972) : Integral Matrices, Academic Press, New York and London.

46. Mitra, S.K. and P. Bhimasankaran (1970a) : Some results on idempotent matrices and a matrix equation connected with the distribution of quadratic forms, Sankhyā, Indian Journal Statistics, Series A, 32, part 3, (353-356).

47. Munn, W.D. and R.Penrose (1955) : A note on inverse Semi- groups, Proc. Cambridge Philos. Soc. 51, (396-399).

48. Munn, W.D. (1961) : Pseudo- inverses in Semigroups, Proc. Cambridge Philos. Soc. 57, (247-250).

49. Mitra, S.K. (1968) : On a generalized inverse of a matrix and applications, Sankhya, Series A, Vol 30, part 1, (107-114).

50. Mitra, S.K. (1968b) : A new class of g-inverse of square matrices, Sankhya, ser. A. 30, (323-330).

51. Mitra, S.K. (1973) : Common solutions to a pair of linear matrix equations, $A_1XB_1 = C_1$, $A_2XB_2 = C_2$, Proc. Cambridge Philos. Soc. (213-216).

52. Mitra, S.K. (1977) : The matrix equation $AXB + CXD = E$, SIAM J. Appl. Math. 32, No. 4, (823-825).

53. Marquardt, D.W. (1969) : Generalized inverses, ridge regression, biased linear estimation and non linear estimation (mimeo report). Engineering Services Division, Engineering Department, E.I. du pont de Nemours and co., Wil mington Delaware.

54. Manherz, R.K. (1968) : Applications of the generalized matrix inverse in network theory. Doctoral dissertation, Northwestern University.

55. Neudecker, H. (1969) : A note on Kronecker Matrix products and Matrix equation systems. SIAM J. Appl. Math. Vol. 17, No. 3, (603-606).

56. Penrose, R. (1955) : *A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 51, (406-413).*

57. Plemmons, R.J. (1971) : *Generalized inverses of Boolean relation matrices, SIAM J. Appl. Math. 20. (426-433).*

58. Pye, W.C. (1980) : *Non negative Drazin inverses, Linear Algebra and its Applications, 30 (149-153).*

59. Pyle, L.D. (1965) : *The generalized inverse in linear programming-Basic theory. Tech. Report, 1, Computer Science Department, Purdue University, Lafayette, Ind.*

60. Rohde, C.A. (1968) : *Special application of the theory of generalized matrix inversion to statistics, Proc. Symp. Theo. Appl. Generalized Inverses of Matrices, Mathematic Ser. No. 4, Texas Technological college Lubbock.*

61. Rohde, C.A. (1965) : *Generalized inverses of partitioned matrices, J. Soc. Indust. Appl. Math. 13, No. 4, (1033-1035).*

62 Rohde, C.A. (1966) : *Some results on generalized inverses, SIAM Review, Vol. 8, No. 2, (201-205).*

63. Rado, R. (1956) : *Note on generalized inverses of matrices Proc. Cambridge Philos. Soc. 52, (600-601).*

64. Rotman, J. (1965) : The Theory of Groups, Allyn and Bacon, Inc. Boston.

65. Rao, C.R. (1962) : A note on a generalized inverse of a matrix with applications to problems in mathematical statistics, J.R. Statist. Soc. B., Vol. 24, (152-158).

66. Rao, C.R. (1966) : Generalized inverse for matrices and its applications in Mathematical statistics, Research papers in statistics, Festschrift for J. Neyman, Wiley, New York.

67. Rao, C.R. and S.K. Mitra (1971) : Generalized inverses of matrices and its applications, John Wiley and Sons, Inc. New-York.

68. Rao, C.R. (1955) : Analysis of dispersion for multiply classified data with unequal numbers in cells Sankhyā, 15, (253-280).

69. Rutherford, D.E. (1963) : Inverses of Boolean matrices, Pro. Glasgow Math. Assoc. 6, (49-53).

70. Rosen, J.B. (1960) : The gradient projection method for nonlinear programming, part I : Linear constraints, SIAM J. Appl. Math. , 8, (181-217).

71. Rosen, J.B. (1961) : The gradient projection method for nonlinear programming, part II : Nonlinear constraints, SIAM J. Appl. Math. 9, (514-532).

72. Rong, Guang Hao (1982) : The error bound of the perturbation of the Drazin inverse. Linear Algebra and its Appl. 47, (159-168).

73. Stoll, R.R. and E.T. Wong (1968) : Linear Algebra, Academic Press International Edition, New York and London.

74. Shinozaki, N. and M. Sibuya (1974) : The Reverse order law $(AB)^- = B^-A^-$, Linear Algebra and its Applications, 9, (29-40).

75. Searle, S.R. (1971) : Linear Models, John Wiley, New York, Section 1.7.

76. Searle, S.R. (1966) : Matrix algebra for the Biological Sciences (including applications in Statistics), Wiley, New York.

77. Schenkman, E. (1965) : Group Theory, Van Nostrand, East-West Student Edition-1971.

78. Shanbag, D.N. (1968) : Some remarks on Khatri's result in quadratic forms, Biometrika, 55, (593-595).

79. Shanbag, D.N. (1970) : On the distribution of a quadratic form, Biometrika, 57,(222-223).

80. Sharpe, G.E. and G.P.H. Styan (1965a) : A note on the generalized network inverse, IEEE Transactions on Circuit Theory, 12, (632-633).

81. Sharpe, G.E. and G.P.H. Styan (1965b) : Circuit duality and the general network inverse, IEEE Transactions on Circuit Theory, 12,(22-27).

82. Seshu, S. and M.B. Reed (1961) : Linear Graphs and Electrical Networks. Addison-Wesley, Reading, Mass.

83. Tseng, Y. (1949a) : Generalized inverses of unbounded operators between two unitary spaces, Dokl. Akad. Nauk, SSSR (N.S.), 67, (431-434).

84. Tseng, Y. (1949b) : Properties and classifications of generalized inverses of closed operators, Dokl. Akad. Nauk, USSR, 67,(607-610).

85. Tseng, Y. (1956) : Virtual solutions and generalized inversions, Uspehi Math. Nauk (N.S.),11, (213-215).

86. Vertuba, C. (1963) : Generalized inverses and singular equations in functional analysis. Doctoral dissertation, University of Michigan.